Notation: If R is binary relation on a set A and if $(a, b) \in R$, then I'll say a is related to b. If $(b, a) \in R$, then I'll say b is related to a

Equivalence Relations

Definition 1. Let A be a set and let R be a binary relation on A, which is both reflexive, transitive, and **symmetric**. Then R is called an equivalence relation on A, and it is denoted by $\sim$.

For example, if $(a, b) \in R$, then this means that a is related to b under the relation R. For this reason we use the notation $a \sim b$ to indicate that $(a, b) \in R$. So, $a \sim b$ is equivalent to $(a, b) \in R$. Once again instead of R, we'll write $\sim$.

Definition 2. Let $\sim$ be an equivalence relation on A and let $a \in A$. Then the equivalence class of a (denoted by $\overline{a}$) is defined to be all the elements related to a. To find this equivalence class, you look for all the ordered pairs in $\sim$ which start with a and you take the second element of the ordered pair. Notice that there is no empty equivalence class, because if $a \in A$, then $a \sim a$, because $\sim$ is reflexive. Remember R is reflexive iff (a, a) is in R for every a in A. This means $a \sim a$. Thus, a is in the equivalence class of a.

Example 3. Let $A = \{1, 2, 3\}$ and $\sim = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$. Then $\sim$ is an equivalnce relation on A. The equivalence classes are:

$\overline{1}$={ all elements related to 1 }=$\{1, 2\}$.

$\overline{2}$={ all elements related to 2 }=$\{2, 1\}$.

So, $\overline{2} = \overline{1}$. Remember the order within the set is not important.

$\overline{3}$={ all elements related to 3 }=$\{3\}$.

Notice that the equivalence class of 3 is disjoint from the equivalence class of 1 and from the equivalence class of 2.

Notice also that the union of the equivalence classes is equal to A.

Facts:

(1) If a and b are any two elements of A, then their equivalence classes are either equal or disjoint.

(2) If $a \sim b$, then their equivalence classes are equal.

(3) The equivalence classes partition A.

Partial Orders

Definition 4. Let A be a set and let R be a binary relation on A, which is both reflexive, transitive, and **antisymmetric**. Then R is called a partial order on A, and it is denoted by $\preceq$. The set $(A, \preceq)$ is called a partially ordered set (poset).

For example, if $(a, b) \in R$, then this means that a is related to be under the relation R. For this reason, we use the notation $a \preceq b$ to indicate that $(a, b) \in R$. So, $a \preceq b$ is equivalent to $(a, b) \in R$. Once again instead of R, we'll write $\preceq$.

Example 5. Take A to be the set of real numbers $\mathbb{R}$, and take $\preceq$ to be $\leq$, then $\leq$ is a partial order on $\mathbb{R}$ and $(\mathbb{R}, \leq)$ is a poset.

Example 6. Let S be any nonempty set. Take A to be the power set of S, and take $\preceq$ to be $\subseteq$, then $\subseteq$ is a partial order on A and $(A, \subseteq)$ is a poset. Notice that a and b are related (i.e., (a, b) is in R) if and only if $a \subseteq b$. To prove that $\subseteq$ is a partial order, we need to show that $\subseteq$ is reflexive, transitive, and antisymmetric on A. To show it is reflexive, we need to prove that a and a are related under the relation $\subseteq$. Notice that a itself is a set here. Now to show that a is related to a here means that we need to show that $a \subseteq a$. But, this is always true. So, our relation is reflexive. Now to show that it is transitive, we need to show that for any elements a, b, and c of A, if a is related to b and b is related to c, then a is related to c. Once again keep in mind here that both a, b, and c are sets. But now a is related to b means $a \subseteq b$ and b is related to c means $b \subseteq c$. Thus, we have $a \subseteq b \subseteq c$, which implies $a \subseteq c$. Thus, a is related to c.

Finally, we need to show that our relation is antisymmetric. This means that we want to show if a is related to b and b is related to a, then $a = b$. This must hold for any a and any b in A. But, a is related to b means $a \subseteq b$ and b is related to a means that $b \subseteq a$. Thus, we have that $a \subseteq b$ and $b \subseteq a$. Therefore $a = b$.

NOTE: Let S be a nonempty set. If T is any subset of the power set of S, then $(T, \subseteq)$ is a poset.

Question: **Is it possible to have a binary relation which is both equivalence and a partial order? If yes, then give an example and explain the nature of such a binary relation.**

Answer: Let A be any nonempty set and let R be a binary relation on A. In order for R to be both equivalence and a partial order, we need R to be both reflexive, transitive, symmetric, and antisymmetric. But, in order for R to be antisymmetric, if it has an element of the form (a, b), and if $a \neq b$, then it cannot have (b, a). But, if (a, b) is in R and (b, a) is not, then R becomes not symmetric. Thus, R cannot have elements of the form (a, b) if $a \neq b$. Thus, the only elements which R can have are those of the form (a, a). But, in order for R to be reflexive, we need (a, a) to be in R for every element $a \in A$. (Notice that we have not used transitivity.) For example, if $A = \{1, 2, 3\}$, then the only equivalence relation and partial order on A is $\{(1, 1), (2, 2), (3, 3)\}$.

Definition 7. Let $(A, \preceq)$ be a poset and let $a \in A$. Then a is said to be

1. *maximum* if $x \preceq a$ for every x in A.
2. *minimum* if $a \preceq x$ for every x in A.
3. *maximal* if whenver b is an element of A, such that $a \preceq b$, then $b = a$.
4. *minimal* if whenver b is an element of A, such that $b \preceq a$, then $b = a$.

Examples: In all of the following examples, I'll take $\preceq$ to be $\subseteq$. Notice that if A is a set of sets, then $(A, \subseteq)$ is a poset. In this case, an element a of A is maximum if it contains all other elements of A and it is a maximal if it is not a subset of any other element of A. It is mimimum if it is (I mean a) a subset of all elements of A and it is minimal if no element of A is a subset of a. Notice that every minimum is a minimal and every maximum is a maximal, but the opposite is not necessarily true. Notice also that a poset can have both minimum and maximal or just one of them or none. Notice also that if the maximum exists, then it is unique and if the minimum exists, then it is also unique. Notice also that an element can be both minimal and maximal at the same time. Also a minimal element does not have to be unique. Similarly, a maximal element is not necessarily unique.

Question: Can an element be both minimum and maximum at the same time. If yes, then when?

1. Maximum but no minimum: Consider the poset $(A, \subseteq)$, where $A = \{\{a\}, \{b\}, \{a, b\}\}$. Then $\{a, b\}$ is maximum because it contains all other elements of A (namely $\{a\}$, $\{b\}$). But, there is no minimum because no element of A is a subset of all other elements. Notice also that $\{a, b\}$ is maximal and each of $\{a\}$ and $\{b\}$ is minimal, because no element of A (other than $\{a\}$ itself) is contained in $\{a\}$ and no element of A (other than $\{b\}$ itself) is contained in $\{b\}$.
2. Minimum but no maximum: Consider the poset $(A, \subseteq)$, where $A = \{\{a\}, \{a, b\}, \{a, c\}\}$. Then $\{a\}$ is both minimum and minimal, but A has no maximum. Notice also that both of $\{a, b\}$ and $\{a, c\}$ are maximal.
3. Minimum and maximum: Consider the poset $(A, \subseteq)$, where $A = \{\{a\}, \{a, b\}, \{a, b, c\}\}$. Then $\{a\}$ is both minimum and minimal, and $\{a, b, c\}$ is both maximum and maximal. Notice that $\{a, b\}$ is neither minimal nor maximal (and of course neither minimum nor maximum).
4. No minimum and no maximum: Consider the poset $(A, \subseteq)$, where $A = \{\{a\}, \{b\}, \{a, c\}\}$. Here we have no minimum and no maximum. But, $\{a\}$ and $\{b\}$ are both minimal and $\{b\}$ and $\{a, c\}$ are both maximal. Thus, $\{b\}$ is both minimal and maximal.

5. Consider the poset $(A, \subseteq)$, where $A = \{\{1\}, \{1,2\}\}$. Here $\{1\}$ is both minimum and minimal and $\{1,2\}$ is both maximum and maximal.
6. Consider the poset $(A, \subseteq)$, where $A = \{\{1,2\}, \{1,3\}\}$. Here there is no minimum and no maximum and $\{1,2\}$ and $\{1,3\}$ are both minimal and maximal.

Fact: Let $(A, \preceq)$ be a poset and let $a \in A$. If a is minimum/maximum, then it is also minimal/maximal and it is the only minimal/maximal.

Definition 8. Let $(A, \preceq)$ be a poset and let a, $b \in A$. Then a and b are called *comparable* iff either $a \preceq b$ or $b \preceq a$.

Example 9. Consider the poset $(A, \subseteq)$, where $A = \{\{1\}, \{2\}, \{2,3\}\}$. Then $\{2\}$ and $\{2,3\}$ are comparable, because the first is a subset of the second. On the other hand, $\{1\}$ and $\{2\}$ are not comparable, because neither one is a subset of the other.

Definition 10. Let $\preceq$ be a partial order on A. If for every a and b in A, we have either $a \preceq b$ or $b \preceq a$, then $\preceq$ is called a total order and $(A, \preceq)$ is called a totally ordered set.

Example 11. $(A, \subseteq)$, where $A = \{\{1\}, \{1,2\}, \{1,2,3\}\}$ is a total order.

Example 12. $(A, \subseteq)$, where $A = \{\{1\}, \{2\}, \{1,3\}\}$ is not a total order, because neither one of $\{1\}$ and $\{2\}$ is a subset of each other.

Notation: The greatest lower bound of a and b is denoted by $a \wedge b$ and the least upper bound of a and b is denoted by $a \vee b$.

Example 13. Let S be any nonempty set, let A be the power set of S and let $a \in A$ and $b \in A$. Then $a \wedge b = a \cap b$ and $a \vee b = a \cup b$.

Example 14. Let A be the set of real numbers and let $a \in A$ and $b \in A$. Then $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

Example 15. Consider the poset $(A, \subseteq)$, where $A = \{\{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}\}$. Then

$\{1,2,3\} \wedge \{1,2,4\} = \{1,2\}$.

$\{1,2,3\} \vee \{1,2,4\}$ does not exist.

EXAMPLE 16. Consider the poset $(A, \subseteq)$, where $A = \{\{2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}$. Then

$\{1,2,3\} \wedge \{1,2,4\} = \{2\}$.

$\{1,2,3\} \vee \{1,2,4\} = \{1,2,3,4\}$.

Question Consider the poset $(A, \subseteq)$, where $A = \{\{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$. Then what is:

(a) $\{1\} \wedge \{2\}$?

(b) (a) $\{1\} \vee \{2\}$?

More to Come (especially on Hasse diagrams