Proof of Stenger's Conjecture on Matrix $I^{(-1)}$ of Sinc Methods

Lixing Han*
Department of Mathematics
University of Michigan–Flint
Flint, Michigan 48502, U.S.A.
and
Jianhong Xu†
Department of Mathematics
Southern Illinois University Carbondale
Carbondale, Illinois 62901, U.S.A.

July 1, 2013

Abstract

In this paper, we prove a conjecture, which was proposed by Frank Stenger in 1997, concerning the localization of eigenvalues of the Sinc matrix $I^{(-1)}$, a problem that is important in both the theory and the practice of Sinc methods. In 2003, Iyad Abu-Jeib and Thomas Shores established a partial answer to this unsolved problem. The techniques they have developed, however, turn out to be the key that finally leads to the settlement here of Stenger's conjecture.

Keywords: Sinc methods, approximation, Sinc matrix, eigenvalue localization, Toeplitz matrix, skew symmetry, generating function
AMS Subject Classification: 15A18, 15A42, 15B05, 15B57, 41A30, 65F15, 65T40

*E–mail: `lxhan@umflint.edu`, Phone: +1 (810) 762-3012, Fax: +1 (810) 766-6880.
†E–mail: `jxu@math.siu.edu`, Phone: +1 (618) 453–6510, Fax: +1 (618) 453–5300.

1 Introduction

For the past three decades, Sinc approximation methods have been successfully used in handling a wide variety of computational problems, which arise from interpolation, numerical differentiation and integration, and numerical solution of ordinary and partial differential equations and integral equations, such as Hilbert transforms, Laplace transform inversion, Wiener-Hopf equations, and so on; see [7, 15, 17] and the references therein. Currently, research on Sinc methods remains "ongoing and vibrant" in the words of F. Stenger, who largely laid the foundations for these methods. For most recent relevant works, see, for example, [2, 3, 10, 11, 14, 17].

One of the key ingredients of Sinc methods is the so-called Whittaker's cardinal function, which is defined as a series in terms of the sinc function

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \ \forall x \neq 0,$$

while $\operatorname{sinc}(0) = 1$. For detailed background material, we refer the reader to [17]. In this paper, we are concerned with the Sinc matrix $I^{(-1)}$, which arises from Sinc indefinite integration and convolution. Specifically, $I^{(-1)} \in \mathbb{R}^{n\times n}$ has a Toeplitz structure, whose (j,k)th entry is given by

$$I^{(-1)}_{j,k} = \frac{1}{2} + s_{j-k},$$

where $s_k = \displaystyle\int_0^k \operatorname{sinc}(x)dx$ for $k = 0, \ \pm 1, \ldots, \ \pm(n-1)$. Clearly, this matrix can be written as

$$I^{(-1)} = \frac{1}{2}ee^T + S, \tag{1}$$

where $e \in \mathbb{R}^n$ is the (column) vector of all ones and where

$$S = \begin{bmatrix} s_0 & -s_1 & -s_2 & \cdots & -s_{n-1} \\ s_1 & s_0 & -s_1 & \cdots & -s_{n-2} \\ s_2 & s_1 & s_0 & \cdots & -s_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_0 \end{bmatrix} \tag{2}$$

is skew symmetric and, again, Toeplitz. Note that $s_0 = 0$. We mention that all the eigenvalues of S are purely imaginary as a consequence of its skew

symmetry.

It is easy to justify that all the eigenvalues of $I^{(-1)}$ lie in the closed right half-plane, see Section 2 as well as [1, 17]. However, as Stenger pointed out in his 1997 paper [16], the validity of relevant Sinc approximations hinges on the assumption that all these eigenvalues are located in the open right half-plane. In light of favorable numerical evidence for each $I^{(-1)}$ of order up to $n = 513$, Stenger conjectured in [16] that this may be the case for all $I^{(-1)}$, regardless how large n is. In his 2011 book [17, p.99], Stenger reiterated the conjecture, quoting favorable numerical evidence for each $I^{(-1)}$ of order up to $n = 1024$.

We provide in this paper a proof that finally settles Stenger's conjecture concerning the eigenvalues of $I^{(-1)}$. Our work is mainly motivated by some recent results of I. Abu-Jeib and T. Shores [1]. In fact, we shall follow the same methodology that has been developed in [1]. For the presentation here to be as self-contained as possible, we shall also include the proofs of some important known results, mainly from [1].

2 Proof of Stenger's Conjecture

We shall consider in what follows a slightly more general situation. Instead of $I^{(-1)}$ as in (1), we define

$$A = A(\omega) = \omega ee^T + S \in \mathbb{R}^{n\times n}, \tag{3}$$

where ω is an arbitrary positive number and S is given as in (2). Obviously, $A(1/2) = I^{(-1)}$. To avoid triviality, we always assume $n \geq 2$.

Our main result can be stated as:

THEOREM 2.1 *For $S \in \mathbb{R}^{n\times n}$ as in (2) and for any $\omega > 0$, the matrix A in (3) has all its eigenvalues in the open right half-plane. In particular, all the eigenvalues of $I^{(-1)}$ are located in the open right half-plane.*

Before proceeding to the proof of the above conclusion, we shall discuss some preparatory results first.

For the time being, let us take

$$B = \omega ee^T + C \in \mathbb{R}^{n\times n},$$

where $\omega > 0$ and C is any skew symmetric matrix. With this broader setting, we have the following lemma, which applies, in particular, to our matrix A in (3) as well as to $I^{(-1)}$.

LEMMA 2.1 ([1, Lemma 2.1] and [17, p.98]) *For any $\omega > 0$, the eigenvalues of the matrix B all lie in the closed right half-plane. Let (λ, u) be an eigenpair of B. Then, λ is on the imaginary axis if and only if $e^T u = 0$, that is, u and e are orthogonal to each other. Moreover, when λ is located on the imaginary axis, (λ, u) is also an eigenpair of C.*

Proof: Let (λ, u) be an eigenpair of B. Since C is skew symmetric, u^*Cu is purely imaginary. From

$$\lambda\|u\|_2^2 = u^*Bu = \omega u^*ee^Tu + u^*Cu = \omega\left|e^Tu\right|^2 + u^*Cu,$$

we see that

$$\mathrm{Re}(\lambda) = \frac{\omega\left|e^Tu\right|^2}{\|u\|_2^2} \geq 0.$$

Besides, $\mathrm{Re}(\lambda) = 0$ if and only if $e^Tu = 0$.

Assume now that $\mathrm{Re}(\lambda) = 0$. Then by $Bu = \lambda u$ and $e^Tu = 0$, we obtain $Cu = \lambda u$. This completes the proof. □

Incidentally, as can be seen from the proof above,

$$\mathrm{Im}(\lambda) = \frac{u^*Cu}{\|u\|_2^2}.$$

In addition, the Cauchy-Schwarz inequality implies here $\mathrm{Re}(\lambda) \leq n\omega$. On a separate note, if $\omega < 0$, then clearly we would have $\mathrm{Re}(\lambda) \leq 0$ instead.

Lemma 2.1 shows that each eigenvector of A, and therefore the specific formation of S, plays a crucial role in determining the location of the corresponding eigenvalue. We comment that if S is replaced by a general skew symmetric Toeplitz matrix, the conclusion in Theorem 2.1 may not be true

as illustrated by the following example.

Consider the 3×3 matrix

$$B = \frac{1}{2}ee^T + \begin{bmatrix} 0 & -c & c \\ c & 0 & -c \\ -c & c & 0 \end{bmatrix},$$

where c is any nonzero real number. One can easily verify that the eigenvalues of B are $3/2$ and $\pm i\sqrt{3}c$, thus not all lying in the open right half-plane. For this case, the eigenvectors corresponding to the purely imaginary eigenvalues are indeed both orthogonal to e.

Given the matrix A in (3), however, we shall prove that $e^T u \neq 0$ for any eigenvector u of A, or equivalently that none of the eigenvalues of A show up on the imaginary axis. This leads to the settlement of Stenger's conjecture.

We also remark that according to Lemma 2.1, the specific value of ω — as long as $\omega > 0$ — turns out to be irrelevant. This remains to be the case for all the results in the sequel.

Next, following [1], we adopt the notion of a generating function $f(\theta)$ of a general Toeplitz matrix $C = [c_{j-k}] \in \mathbb{R}^{n\times n}$, namely, a function $f(\theta)$ such that

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta, \quad k = 0, \ \pm 1, \ldots, \ \pm(n-1).$$

To deal with potential (nonremovable) singularities, all integrals here, and in what follows, are interpreted in the sense of the Cauchy principal value whenever appropriate.

LEMMA 2.2 ([1, Lemma 2.3]) *The matrix S given in (2) has a generating function*

$$f(\theta) = \frac{i}{\theta}.$$

Let

$$T = -iS. \tag{4}$$

Then the generating function of T is given by $f(\theta) = \dfrac{1}{\theta}$.

Proof: By direct calculation using $f(\theta) = \dfrac{i}{\theta}$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{i\cos(k\theta)}{\theta} + \frac{\sin(k\theta)}{\theta}\right)d\theta \\ &= \frac{1}{\pi}\int_{0}^{\pi}\frac{\sin(k\theta)}{\theta}d\theta.\end{aligned}$$

When $k = 0$, the above gives zero, i.e. s_0. When $k > 0$, we use the substitution $k\theta = \pi x$ to arrive at

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta = \frac{1}{\pi}\int_{0}^{\pi}\frac{\sin(k\theta)}{\theta}d\theta = \int_{0}^{k}\operatorname{sinc}(x)dx = s_k.$$

The generating function of T follows directly from that of S. □

We mention that the generating function relevant to $I^{(m)}$, where $m \geq 0$, is known to be $f(\theta) = (i\theta)^m$; see [15]. Specifically, for each $m = 0, 1, 2, \ldots$, the matrix $I^{(m)}$ is defined as $I^{(m)} = [\delta_{j,k}^{(m)}]$, where

$$\delta_{j,k}^{(m)} = h^m \frac{d^m}{dx^m}\left[\operatorname{sinc}\left(\frac{x - jh}{h}\right)\right]\Bigg|_{x=kh}$$

with $h > 0$ being the step size. Lemma 2.2, therefore, can be thought of as an extension of this result. We also mention that in [1], Abu-Jeib and Shores attributed such generating function methodology to an earlier work by W. Trench [19], who employed the Lebesgue integral but provided many of the ideas that we can use here.

REMARK 2.1 For simplicity, we shall focus in our remaining proofs on the matrix T given by (4). This matrix is hermitian, which implies that its eigenvalues must be all real. Clearly, for some real b, (b, u) is an eigenpair of T if and only if (ib, u) is an eigenpair of S.

In [1], it has been established that the following proposition is true for $I^{(-1)}$, which partially solves Stenger's conjecture. We adapt it to the slightly more general case of our matrix A that involves $\omega > 0$. For completeness, we also include its proof below.

THEOREM 2.2 ([1, Theorem 2.4]) *If $\lambda = ib$ is a purely imaginary eigenvalue of the matrix A that is given in (3), then $|\lambda| > \frac{1}{\pi}$.*

Proof: Let u be the eigenvector of A corresponding to λ. Based on Lemma 2.1, the assumption of λ being purely imaginary is equivalent to $e^T u = 0$. Without loss of generality, we assume $b \geq 0$. We prove this theorem by contradiction, i.e. by starting with $0 \leq b \leq \frac{1}{\pi}$.

We claim first that for $\theta \in [-\pi, 0) \cup (0, \pi]$, except at $\theta = -\pi$ and $\theta = \pi$,

$$\left(\frac{1}{\theta} - b\right) \sin(\theta) > 0. \tag{5}$$

Clearly this is true when $b = 0$. When $0 < b \leq \frac{1}{\pi}$, (5) can be easily verified by checking the sign of each factor on the left-hand side over the intervals $[-\pi, 0)$ and $(0, \pi]$, respectively. Notice that $\theta = 0$ is a removable singularity since $\lim_{\theta \to 0} \left(\frac{1}{\theta} - b\right) \sin(\theta) = 1$.

By Lemma 2.1 and Remark 2.1, we know that (b, u) is an eigenpair of T. Let

$$U(z) = u_1 + u_2 z + \ldots + u_n z^{n-1},$$

where $z \in \mathbb{C}$, be the eigenpolynomial[1] associated with u. Due to the assumption $e^T u = 0$, $U(z)$ can be factored as

$$U(z) = (z - 1)\hat{U}(z).$$

Observe now that we can always choose a vector v such that

$$V(z) = v_1 + v_2 z + \ldots + v_n z^{n-1}$$

can be factored as

$$V(z) = (z + 1)\hat{U}(z).$$

[1]Observe, here and in the proof of Theorem 2.1, that such a polynomial $U(z)$ is well-defined and nontrivial, i.e. at least degree one, since $u \neq 0$ and $e^T u = 0$. In particular, the latter condition guarantees that $z = 1$ is a zero of $U(z)$. A similar comment applies to the polynomial $V(z)$ by its definition.

On one hand, we have

$$\begin{aligned}\langle Tu, v\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\theta}U(e^{i\theta})\overline{V(e^{i\theta})}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(e^{i\theta}-1)(e^{-i\theta}+1)}{\theta}\left|\hat{U}(e^{i\theta})\right|^2 d\theta \\ &= \frac{i}{\pi}\int_{-\pi}^{\pi}\frac{\sin(\theta)}{\theta}\left|\hat{U}(e^{i\theta})\right|^2 d\theta. \qquad (6)\end{aligned}$$

On the other hand, we obtain in a similar fashion that

$$\langle bu, v\rangle = \frac{i}{\pi}\int_{-\pi}^{\pi} b\sin(\theta)\left|\hat{U}(e^{i\theta})\right|^2 d\theta. \qquad (7)$$

Because of (6), (7), and $\langle Tu, v\rangle = \langle bu, v\rangle$, it follows that

$$\int_{-\pi}^{\pi}\left(\frac{1}{\theta}-b\right)\sin(\theta)\left|\hat{U}(e^{i\theta})\right|^2 d\theta = 0.$$

This last equality, however, is a contradiction because of (5) along with the fact that $\left|\hat{U}(e^{i\theta})\right|^2 > 0$ except (possibly) at only a finite number of values of θ on the interval $[-\pi, \pi]$. □

Finally, with all the preliminary results at hand, we are ready to present the proof of our Theorem 2.1.

Proof of Theorem 2.1: First of all, according to Lemma 2.1, it suffices to show that the matrix A as in (3) does not have any purely imaginary eigenvalues.

Based on Theorem 2.2, the proof here further reduces to demonstrating that for any real number b such that $|b| > \frac{1}{\pi}$, $\lambda = ib$ cannot be an eigenvalue of A.

We proceed by way of contradiction. Suppose to the contrary that there exists some real number b with $|b| > \frac{1}{\pi}$, such that $\lambda = ib$ is an eigenvalue of A. By Lemma 2.1, this means that its corresponding eigenvector u satisfies

$e^T u = 0$. Besides, Lemma 2.1 and Remark 2.1 imply that (b, u) must be an eigenpair of T. Without loss of generality, we assume that $b > \dfrac{1}{\pi}$.

We first claim that for $\theta \in [-\pi, 0) \cup (0, \pi]$, except at $\theta = \dfrac{1}{b}$,

$$\left(\frac{1}{\theta} - b\right) \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta - \frac{1}{b}}{2}\right) < 0. \tag{8}$$

This can be readily verified by checking the sign of each factor on the left-hand side over the intervals $[-\pi, 0)$, $(0, b^{-1}]$, and $[b^{-1}, \pi]$, respectively. Also observe that $\theta = 0$ is a removable singularity, since

$$\lim_{\theta \to 0} \left(\frac{1}{\theta} - b\right) \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta - \frac{1}{b}}{2}\right) = -\frac{1}{2} \sin\left(\frac{1}{2b}\right).$$

Again, by Lemma 2.1 and Remark 2.1, (b, u) must be an eigenpair of T. Let

$$U(z) = u_1 + u_2 z + \ldots + u_n z^{n-1},$$

where $z \in \mathbb{C}$, be the eigenpolynomial associated with u. Using the assumption $e^T u = 0$, we see that $z - 1$ divides $U(z)$, i.e. $U(z) = (z-1)\hat{U}(z)$. We now choose

$$V(z) = \hat{U}(z)(z - e^{i/b}),$$

which again is a polynomial of degree at most $n - 1$. Suppose that $V(z) = v_1 + v_2 z + \ldots + v_n z^{n-1}$ for some v.

On one hand, noting that

$$U(z)\overline{V(z)} = \left|\hat{U}(z)\right|^2 (z-1)(\overline{z} - e^{-i/b}),$$

we have

$$\begin{aligned}
\langle Tu, v \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\theta} U(e^{i\theta}) \overline{V(e^{i\theta})} d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(e^{i\theta} - 1)(e^{-i\theta} - e^{-i/b})}{\theta} \left|\hat{U}(e^{i\theta})\right|^2 d\theta \\
&= \frac{2e^{-\frac{i}{2b}}}{\pi} \int_{-\pi}^{\pi} \frac{\sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta - \frac{1}{b}}{2}\right)}{\theta} \left|\hat{U}(e^{i\theta})\right|^2 d\theta. \qquad (9)
\end{aligned}$$

On the other hand, we obtain in a similar fashion that

$$\langle bu, v\rangle = \frac{2e^{-\frac{i}{2b}}}{\pi}\int_{-\pi}^{\pi} b\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta-\frac{1}{b}}{2}\right)\left|\hat{U}(e^{i\theta})\right|^2 d\theta. \tag{10}$$

Hence, by (9), (10), and $\langle Tu, v\rangle = \langle bu, v\rangle$, we arrive at

$$\int_{-\pi}^{\pi}\left(\frac{1}{\theta}-b\right)\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta-\frac{1}{b}}{2}\right)\left|\hat{U}(e^{i\theta})\right|^2 d\theta = 0.$$

This last equality, however, leads to a contradiction because of (8) and the fact that $\left|\hat{U}(e^{i\theta})\right|^2 > 0$ except at only a finite number of values of θ on the interval $[-\pi, \pi]$. □

From Lemma 2.1 and Theorem 2.1, we also have

COROLLARY 2.1 *Let the matrix A be defined as in (3) with $\omega > 0$ and S being given by (2). Then, any eigenvector u of A must satisfy $e^T u \neq 0$. In particular, any eigenvector of the matrix $I^{(-1)}$ has this property.*

Furthermore, on denoting $\sigma(X)$ the spectrum of a square matrix $X \in \mathbb{C}^{n\times n}$, namely the set of its eigenvalues, we can claim

COROLLARY 2.2 *Let S and A be defined as in (2) and (3), respectively. Then*

$$\sigma(S) \cap \sigma(A) = \emptyset$$

holds for all $\omega > 0$.

Finally, according to Theorem 2.1, we notice that $0 \notin \sigma(A)$; therefore, it follows:

COROLLARY 2.3 ([1, Corollary 2.5]) *The matrix A given in (3) must be nonsingular for any $\omega > 0$. In particular, $I^{(-1)}$ is nonsingular.*

3 Concluding Remarks

In this work, we make use of the methodology in [1] by Abu-Jeib and Shores to give, for the first time, a complete answer to Stenger's conjecture on the localization of eigenvalues of the Sinc matrix $I^{(-1)}$.

The matrix A in (3), including the Sinc matrix $I^{(-1)}$ as its special case, is rich in structure. It consists of a particular rank one modification to a skew symmetric Toeplitz matrix. S. Friedland, for example, called such a matrix almost skew symmetric in [5], where he derived remarkable inequalities regarding its real eigenvalues. As a matter of fact, the results here have been built largely by exploiting such a very rich structure through the generating function approach.

The matrix S is not only Toeplitz skew symmetric but also the so-called centro skew symmetric. For the symmetric as well as centro symmetric case, A. Cantoni and P. Butler [4] provided some useful characterizations of the associated eigenvalues and eigenvectors. It appears that such results may be extended to the matrix S for us to gain a deeper understanding of its eigen structure. This appears to be an interesting problem for future research.

Speaking of rank one modifications on a general matrix, we mention that there are some fascinating recent results concerning spectral changes via the notion of genericity, see, for example, [6, 8, 9, 12, 13]. In [8, Theorem 2.3], it states that the following is true for "almost" all — in the sense of genericity, i.e. except on a set of measure zero — $X \in \mathbb{C}^{n\times n}$ and $(u, v) \in \mathbb{C}^n \times \mathbb{C}^n$: If

$$\sigma(X + uv^*) \cap \sigma(X) = \emptyset, \tag{11}$$

then all the eigenvalues of $X + uv^*$ are (algebraically) simple.

In connection with the matrix A given in (3), the above conclusion indicates that the eigenvalues of A are likely all simple as Corollary 2.2 gives exactly the condition in (11). If this happens to be the case, A will consequently be diagonalizable. In fact, Stenger raised a separate question in [16] as to whether the Sinc matrix $I^{(-1)}$ is diagonalizable. Similarly, we can ask the question here as to whether, more generally, the matrix A is diagonalizable, regardless what positive value ω may take. This appears to be another interesting problem for future research. We mention that in [1], it has been

shown that the eigenvalues of S are all simple. This, according to the well-known Bauer-Fike theorem [18, p.192], implies that all the eigenvalues of A are also simple as long as ω is sufficiently close to zero. Consequently, the matrix A is diagonalizable for sufficiently small values of ω.

Finally, we remark that the matrix $I^{(-1)}D$, where D is a diagonal matrix having a positive diagonal, arises in several applications of Sinc methods. In the same spirit as the proof of our Lemma 2.1, it is straightforward to show that all the eigenvalues of $I^{(-1)}D$ lie in the closed right half-plane; also see [17, p.98]. It is still an open problem, however, as to whether these eigenvalues – or, more generally, the eigenvalues of AD – are all located within the open right half-plane, just as those of A. We think that such a question, too, deserves further investigation.

Acknowledgments

The authors thank the anonymous reviewers for their careful reading of this paper and useful comments, which have helped improve the presentation of the material. In particular, the matrix $I^{(-1)}D$ in our concluding remarks was brought up by one of the reviewers.

References

[1] I. Abu-Jeib and T. Shores, On properties of matrix $I^{(-1)}$ of Sinc methods, *New Zealand Journal of Mathematics*, 32: 1–10, 2003.

[2] N. Alonso III and K. Bowers, The symmetric Sinc-Galerkin method yields ADI model problems, *Applied Mathematics and Computation*, 219: 3847–3854, 2012.

[3] G. Baumann and F. Stenger, Fractional calculus and Sinc methods, *Fractional Calculus and Applied Analysis: An International Journal for Theory and Applications*, 14: 568–622, 2011.

[4] A. Cantoni and P. Butler, Eigenvalues and eigenvectors of symmetric centrosymmetric matrices, *Linear Algebra and Its Applications*, 13: 275–288, 1976.

[5] S. Friedland, Eigenvalues of almost skew symmetric matrices and tournament matrices, *Combinatorial and Graph-Theoretical Problems in Linear Algebra: The IMA Volumes in Mathematics and Its Applications*, 50: 189–206, 1993.

[6] L. Hörmander and A. Melin, A remark on perturbations of compact operators, *Mathematica Scandinavica*, 75: 255–262, 1994.

[7] J. Lund and K. Bowers, *Sinc Methods for Quadrature and Differential Equations*, SIAM, Philadelphia, 1992.

[8] C. Mehl, V. Mehrmann, A. Ran, and L. Rodman, Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations, *Linear Algebra and Its Applications*, 435: 687–716, 2011.

[9] J. Moro and F. Dopico, Low rank perturbation of Jordan structure, *SIAM Journal on Matrix Analysis and Applications*, 25: 495–506, 2003.

[10] T. Okayama, T. Matsuo, and M. Sugihara, Sinc-collocation methods for weakly singular Fredholm integral equations of the second kind, *Journal of Computational and Applied Mathematics*, 234: 1211–1227, 2010.

[11] K. Parand, M. Dehghan, and A. Prikhedri, The Sinc-collocation method for solving the Thomas-Fermi equation, *Journal of Computational and Applied Mathematics*, 237: 244–252, 2013.

[12] S. Savchenko, Typical changes in spectral properties under perturbations by a rank-one operator, *Mathematical Notes*, 74: 557–568, 2003.

[13] S. Savchenko, On the change in the spectral properties of a matrix under perturbations of sufficiently low rank, *Functional Analysis and Its Applications*, 38: 69–71, 2004.

[14] W. Slemp and R. Kapania, Imposing boundary conditions in Sinc method using highest derivative approximation, *Journal of Computational and Applied Mathematics*, 230: 371–392, 2009.

[15] F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions*, Springer, New York, 1993.

[16] F. Stenger, Matrices of Sinc methods, *Journal of Computational and Applied Mathematics*, 86: 297–310, 1997.

[17] F. Stenger, *Handbook of Sinc Numerical Methods*, CRC Press, Boca Raton, 2011.

[18] G. Stewart and J. Sun, *Matrix Perturbation Theory*, Academic Press, Boston, 1990.

[19] W. Trench, Some special properties of hermitian Toeplitz matrices, *SIAM Journal on Matrix Analysis and Applications*, 15: 938–942, 1994.