

SOME PROPERTIES OF COMMUTING AND ANTI-COMMUTING m -INVOLUTIONS

NOTE: THIS VERSION CONTAINS APPENDICES NOT INCLUDED IN THE ORIGINAL PUBLICATION

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Abstract. We define a m -involution to be a matrix $K \in \mathbb{C}^{n \times n}$ for which $K^m = I$. In this paper, we investigate the class $S_m(A)$ of m -involutions that commute with a diagonalizable matrix $A \in \mathbb{C}^{n \times n}$. A number of basic properties of $S_m(A)$ and its related subclass $S_m(A, X)$ are given, where X is an eigenvector matrix of A . Among them, $S_m(A)$ is shown to have a torsion group structure under matrix multiplication if A has distinct eigenvalues and have non-denumerable cardinality otherwise. The constructive definition of $S_m(A, X)$ allows one to generate all m -involutions commuting with a matrix with distinct eigenvalues. Some related results are also given for the class $\tilde{S}_m(A)$ of m -involutions that anti-commute with a matrix $A \in \mathbb{C}^{n \times n}$.

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1. Introduction. Let J represent the *exchange matrix* of order n , defined by $J_{i,j} = \delta_{i,n-j+1}$ for $1 \leq i, j \leq n$ where $\delta_{i,j}$ is the Kronecker delta. If $A \in \mathbb{C}^{n \times n}$ commutes with J , then A is called *centrosymmetric*. Centrosymmetric matrices, which appear in numerous applications, include the class of symmetric Toeplitz matrices and the class of bisymmetric matrices. A number of papers (among them, [1], [2], [6], [7], and [10]) have investigated the generalization where J is replaced by an involutory matrix K .

More recently, W. F. Trench (in [8] and [9]) investigated the set of complex matrices A that satisfy $AK = \zeta^j KA$, where ζ is an m -th root of unity, $0 \leq j \leq m-1$, and K 's minimal polynomial is $x^m - 1$ for an integer $m > 1$. Trench referred to such K as m -involutions, but in this paper we use the term *m -involution* to refer to any matrix K for which $K^m = I$. We refer to the set of matrices whose minimal polynomial is $x^m - 1$ as the *non-trivial m -involutions*. This terminology is consistent with usage in the $m = 2$ case, where the matrices $\pm I$ are regarded as trivial involutions.

In this paper, we define a *K_m -symmetric* matrix to be a complex matrix A that commutes with an m -involutory matrix K . In papers that have studied K_2 -symmetric and K_m -symmetric matrices, the matrix K is usually considered fixed and the properties of the matrix A are studied. One difficulty in applying results of this nature is that except in special cases, it's typically not easy to discern the K_m -symmetry of a matrix A by mere inspection. In the current paper, we take a somewhat different approach by fixing A and studying the class $S_m(A)$ of m -involutory matrices K that commute with it. To begin the investigation, we introduce a constructive subset of $S_m(A)$ that enables the easy generation of m -involutions that commute with A . Now the task of identifying the K_m -symmetry of a matrix A can, in some cases, be reduced to a relatively straightforward computation. We also show that $S_m(A)$ has a torsion group structure under matrix multiplication if A has distinct eigenvalues and is infinite otherwise.

If $A \in \mathbb{C}^{n \times n}$ anti-commutes with the exchange matrix J , then A is called *skew-centrosymmetric* or *centroskew* per [4]. As in the centrosymmetric case, the papers [6], [7], and [10] investigated the case where J is replaced by a general involution, while

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[8] looked at a broader generalization (mentioned above) that includes the commuting and anti-commuting cases. This motivated a study of the anti-commuting case, and so we close this paper by establishing some related results for the class $\tilde{S}_m(A)$ of m -involutory matrices K which anti-commute with A .

2. Preliminaries. In [6], [7], and [10], several K_2 -symmetric matrix analogs to well-known results for centrosymmetric matrices were established. For example, during the 1960s and 1970s, Collar [4], Andrew [2], and Cantoni and Butler [3] each noted (in slightly different contexts) that the eigenbasis $\{x_i\}_{i=1}^n$ of a $n \times n$ centrosymmetric matrix A is composed of $\lceil \frac{n}{2} \rceil$ *symmetric eigenvectors* (i.e., $x_i = Jx_i$) and $\lfloor \frac{n}{2} \rfloor$ *skew-symmetric eigenvectors* (i.e., $x_i = -Jx_i$). In [6], this observation was extended to the situation where J was replaced by a non-trivial involutory matrix K and both A and K are real symmetric ([5, Lemma 3.11]). This was a direct generalization of the context considered in [3]. The paper [10] extended this to the Hermitian case for A and K ([8, Propositions 3.5 and 4.1]). In [7], W. F. Trench strengthened this result ([6, Theorem 7]) by showing that K_2 -symmetric matrices A have a basis consisting of K -*symmetric* (i.e., $x_i = Kx_i$) and K -*skew-symmetric eigenvectors* (i.e., $x_i = -Kx_i$) without the Hermitian condition assumed in [10]. Trench also established the converse, thereby generalizing a result of Andrew for centrosymmetric matrices ([2, Theorem 2]). We will refer to these two results of Trench collectively as the *Eigenbasis Theorem* for K_2 -symmetric matrices.

More recently, in [8], Trench extended this result to the class of complex matrices A which commute with a non-trivial m -involutory matrix K . Let $\zeta = e^{2\pi i/m}$. His result ([7, Theorem 13]) states that if $K \in \mathbb{C}^{n \times n}$ is non-trivial m -involutory, $A \in \mathbb{C}^{n \times n}$ is K_m -symmetric, and λ is an eigenvalue of A , then the λ -eigenspace of A has a basis in the union $Q_A = \bigcup_{j=0}^{m-1} \{x | Kx = \zeta^j x\}$. Conversely, he showed that if a matrix A has n linearly independent vectors in Q_A , then A is K_m -symmetric. Accordingly, we will collectively refer to these two more general statements as the *Eigenbasis Theorem* for K_m -symmetric matrices. Following Trench, we will refer to vectors satisfying $Kx = \zeta^r x$ as (K, r) -*symmetric* vectors.

In this paper, we use the notation $S_m(A)$ to denote the set of complex m -involutory matrices K that commute with a complex matrix A . We use the notation $T_p(x)(A)$ to denote the set of complex m -involutory matrices with minimal polynomial $p(x)$ that commute with A . Note that

$$S_m(A) = \bigcup_{\substack{p(x) | x^m - 1 \\ p(x) \in \mathbb{C}[x] \\ p(x) \text{ monic}}} T_p(x)(A).$$

Let $\alpha, \mu \in \{0, 1, \dots, k-1\}$ and let R and S be non-trivial m -involutions. In Trench's papers [8] and [9], the class of matrices A satisfying the equation

$$(1) \quad RAS^{-\alpha} = \zeta^\mu A$$

are studied. When $\alpha = 1$, $\mu = 0$, and $R = S$, the matrices R and S belong to the set $T_{p(x)}(A)$ where $p(x) = x^m - 1$. Trench's focus is on the class of matrices A satisfying (1), and while he does provide some general results concerning non-trivial m -involutions, he does not attempt to investigate the collective properties of either the set $T_{p(x)}(A)$ or its superset $S_m(A)$.

3. Construction and Properties of $S_m(A, X)$. For what follows, we assume that A is diagonalizable. Let $A = X\Lambda X^{-1}$ where X is an eigenvector matrix of A and Λ is the diagonal eigenvalue matrix of A . For convenience, we will assume that each of the columns of X are normalized and that there is a fixed ordering for the eigenvalues in Λ .

If we wish to construct a non-trivial subset of $S_m(A)$, consideration of simultaneous diagonalization and familiarity with the Eigenbasis Theorem for K_m -symmetric matrices leads naturally to an investigation of matrices of the form $K = XDX^{-1}$, where D is a diagonal matrix whose diagonal elements belong to the set of m -th roots of unity.

DEFINITION 3.1. *For a fixed eigenvector matrix X of $A \in \mathbb{C}^{n \times n}$, we define*

$$S_m(A, X) = \left\{ XDX^{-1} \mid D \text{ diagonal with } D_{i,i} \in \{\zeta^j\}_{j=0}^{m-1} \right\}$$

where $\zeta = e^{2\pi i/m}$.

The following theorem lists some of the more elementary properties concerning the set $S_m(A, X)$ and its relationship to $S_m(A)$.

THEOREM 3.2. *For a fixed eigenvector matrix X of $A \in \mathbb{C}^{n \times n}$, we have:*

$$(2) \quad S_m(A, X) \subseteq S_m(A),$$

$$(3) \quad |S_m(A, X)| = m^n,$$

and

$$(4) \quad S_2(A) = \bigcup_X S_2(A, X)$$

where the union ranges over all possible eigenvector matrices X of A .

Proof. (2) and (3) follow easily from the definition of $S_m(A, X)$.

To show (4), we first note that the trivial involutions $\pm I$ in $S_2(A)$ are obtained using $D = \pm I$ for any eigenvector matrix X of A . For any non-trivial involution K , the Eigenbasis Theorem for K_2 -symmetric matrices states that there exists an eigenbasis for A consisting of K -symmetric and K -skew-symmetric eigenvectors. Let $X = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}$ have columns comprised of such a basis. If we choose $D = (d_{i,j})_{1 \leq i,j \leq n}$ to be the diagonal matrix having -1 at element $d_{j,j}$ for each K -skew-symmetric eigenvector x_j and 1 for the remaining diagonal elements, then $K = XDX^{-1}$. That is, we recover K with this choice of D .

So, for any non-trivial involutory $K \in S_2(A)$ that there exists an eigenvector matrix X of A and choice of D such that $K = XDX^{-1}$. This shows that $S_2(A) \subseteq \bigcup_X S_2(A, X)$, and (2) shows the reverse inclusion. $\square$

In the proof of the third assertion of Theorem 3.2, we showed how one could choose the elements of a diagonal matrix D to recover a matrix $K \in S_2(A)$, given an eigenvector matrix for A consisting of K -symmetric and K -skew-symmetric columns. We now state the analagous “recovery” theorem for a nontrivial m -involution $K \in S_m(A)$. The proof follows along the same lines as that for the $S_2(A)$ case so we omit it.

THEOREM 3.3. *Let $A \in \mathbb{C}^{n \times n}$ be K_m -symmetric with K a non-trivial m -involution. Suppose $X = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}$ is an eigenvector matrix for A , where*

each column x_j is (K, r_j) -symmetric for some $0 \leq r_j \leq m-1$. Since A is K_m -symmetric, such an X is guaranteed to exist by the Eigenbasis Theorem for K_m -symmetric matrices. Then there exists a diagonal matrix $D = (d_{i,j})_{1 \leq i,j \leq n}$ such that $K = XDX^{-1}$ and whose diagonal elements $d_{j,j}$ satisfy $d_{j,j} = \zeta^{n-r_j}$. As usual, $\zeta = e^{2\pi i/m}$.

The following corollary is immediate from Theorem 3.3.

COROLLARY 3.4. *Let $A \in \mathbb{C}^{n \times n}$ be K_m -symmetric where K is a non-trivial m -involution. If X_a and X_b are eigenvector matrices of A , each of whose columns is (K, r_j) -symmetric for some $0 \leq r_j \leq m-1$, then $K \in S_m(A, X_a) \cap S_m(A, X_b)$*

Example. If A is a centrosymmetric matrix and X is any eigenvector matrix of A comprised of symmetric and skew-symmetric vectors, then the exchange matrix J is guaranteed to be found in $S_2(A, X)$.

Equations (2) and (3) of Theorem 3.2 show that after one determines an eigenvector matrix X and its inverse X^{-1} for a diagonalizable matrix A , one can then identify m^n different K_m -symmetries of A by simply computing the matrix products XDX^{-1} where one varies the diagonal elements of the matrix D over the different combinations of m -th roots of unity. This computation can be further made more efficient by making the following observations.

1. Let $K = XDX^{-1}$ and $\tilde{K} = X\tilde{D}X^{-1}$ be members of $S_m(A, X)$, where the diagonal matrices $D = \text{diagonal}(d_{ii})$ and $\tilde{D} = \text{diagonal}(\tilde{d}_{ii})$ differ only by a factor of ζ in their k th diagonal element. If $\tilde{d}_{kk} = \zeta d_{kk}$, and we denote the i, j element of X^{-1} by y_{ij} , then element $\tilde{k}_{ij}$ of $\tilde{K}$ is related to element k_{ij} of K by

$$\tilde{k}_{ij} = k_{ij} + (\zeta - 1)d_{kk}x_{ik}y_{kj}.$$

So, as one iterates through the products XDX^{-1} by varying D , if an update to D only involves a ζ -scaling of a single diagonal element, then deriving an updated $\tilde{K}$ from K requires just $3n^2$ multiplications and $n^2 + 1$ additions.

2. If $K \in S_m(A, X_j)$, then $\zeta^r K \in S_m(A, X_j)$ for $0 \leq r \leq m-1$. To avoid the unnecessary computation of these scalar multiples of K , we can simply fix a diagonal element of D at 1 (say d_{11}) and vary the other diagonal elements over the m -th roots of unity. To see this, let $K = XDX^{-1}$ be an element of $S_m(A, X_j)$ for some $D = \text{diagonal}(\zeta^{r_1} \quad \zeta^{r_2} \quad \dots \quad \zeta^{r_n})$. Then

$$\zeta^{-r_1} KX = \begin{pmatrix} x_1 & \zeta^{r_2-r_1}x_2 & \dots & \zeta^{r_n-r_1}x_n \end{pmatrix}.$$

This shows that there exists a matrix $\tilde{K} \in S_m(A, X_j)$ such that $K = \zeta^r \tilde{K}$ where $\tilde{K} = X\tilde{D}X^{-1}$ and $\tilde{D}$ has a value of 1 for its $d_{1,1}$ element. Employing this observation reduces the number of matrix product computations needed to enumerate the elements of $S_m(A, X_j)$ down by a factor of m .

At this point, after generating the elements of $S_m(A, X)$, one can then select distinguished elements K from the set and apply known theoretical results for K_m -symmetric matrices to a particular study involving A .

Since $S_m(A, X)$ is finite, it is not surprising that one can make some fairly strong statements about its elements. The next theorem lists two such statements and shows that when a matrix $A \in \mathbb{C}^{n \times n}$ has distinct eigenvalues that these statements apply to $S_m(A)$ itself.

THEOREM 3.5. *If $A \in \mathbb{C}^{n \times n}$ is diagonalizable with an eigenvector matrix X , then*

1. If A has distinct eigenvalues (hence, is diagonalizable), then

$$(5) \quad S_m(A) = S_m(A, X)$$

2. The number of non-trivial m -involutory matrices in $S_m(A, X)$ is

$$(6) \quad \sum_{j=0}^m (-1)^j \binom{m}{j} (m-j)^n.$$

Proof. The proof of the first assertion (5) is elementary and has been omitted.

For the second assertion (6) of Theorem 3.5, we wish to compute the cardinality of $T_p(x)(A)$ when the minimal polynomial $p(x)$ equals $x^m - 1$ and has distinct roots. To determine this, one simply needs to count the number of diagonal matrices D whose set of diagonal elements contain at least one of each of the m -th roots of unity. This can be accomplished by a standard inclusion-exclusion enumeration. $\square$

We now characterize the algebraic structure of $S_m(A, X)$, which as we've shown in Theorem 3.5, equals $S_m(A)$ when A has distinct eigenvalues.

THEOREM 3.6. *Let $A \in \mathbb{C}^{n \times n}$ be diagonalizable with an eigenvector matrix X . Then the set $S_m(A, X)$ under matrix multiplication forms a torsion group $G_m(A, X)$ that is isomorphic to $\bigoplus_{i=1}^n \mathbb{Z}_m$.*

Proof. That $S_m(A, X)$ is an abelian group follows easily from its definition. The mapping

$$X \begin{pmatrix} \zeta^{j_1} & & 0 \\ & \ddots & \\ 0 & & \zeta^{j_n} \end{pmatrix} X^{-1} \mapsto (j_1) \oplus \dots \oplus (j_n)$$

is clearly a group homomorphism from $G_m(A, X)$ to $\bigoplus_{i=1}^n \mathbb{Z}_m$ with a trivial kernel, thereby establishing the isomorphism. $\square$

COROLLARY 3.7. *If A is centrosymmetric with distinct eigenvalues, then the involutory matrices that commute with it are centrosymmetric.*

Proof. This follows from equation (5) and Theorem 3.6 since the exchange matrix J is a member of the abelian group $G_2(A, X)$. $\square$

It's easy to produce examples that show that Corollary 3.7 fails if the restriction that A has distinct eigenvalues is violated. However, it's also not hard to show that Corollary 3.7 holds without the restriction to involutory matrices. We start by defining the simultaneously diagonalizable family $R(A, X)$.

DEFINITION 3.8. *For a fixed eigenvector matrix X of $A \in \mathbb{C}^{n \times n}$, let*

$$R(A, X) = \{XDX^{-1} | D \in \mathbb{C}^{n \times n} \text{ diagonal}\}.$$

It's clear that $R(A, X)$ is an superset of $S_m(A, X)$ for every m , and that $R(A, X)$ is a commutative monoid under matrix multiplication.

LEMMA 3.9. *Suppose $A \in \mathbb{C}^{n \times n}$ has distinct eigenvalues. If $B \in \mathbb{C}^{n \times n}$ commutes with A , then $B \in R(A, X)$.*

Proof. Since $\Lambda = X^{-1}AX$ has distinct entries and B commutes with A , it follows that $X^{-1}BX$ commutes with Λ . The only matrices which commute with diagonal matrices with distinct entries are diagonal matrices. Therefore $X^{-1}BX$ is diagonal, and so $B \in R(A, X)$. $\square$

THEOREM 3.10. *If A is centrosymmetric with distinct eigenvalues, then the matrices that commute with it are centrosymmetric.*

Proof. Suppose matrices B_i and B_j commute with A . From Lemma 3.9, we have that B_i commutes with B_j . Letting $B_i = J$ shows that B_j is centrosymmetric. $\square$

COROLLARY 3.11. *If A is real, bisymmetric and has distinct eigenvalues, then the matrices that commute with it are bisymmetric.*

Proof. If A has real entries and $A = A^T$, then X can be taken to be a matrix with real entries and $X^{-1} = X^T$. If B commutes with A , then $X^T B X$ must be a diagonal matrix from Lemma 3.9. Hence

$$X^T B^T X = (X^T B X)^T = D^T = D = X^T B X.$$

Multiplying on the left by X and the right by X^T gives $B^T = B$. This and Theorem 3.10 yield the result. $\square$

4. $S_m(A, X)$ and the Non-Distinct Eigenvalue Case. When a diagonalizable matrix A has distinct eigenvalues, equation (5) implies that Theorem 3.6 is a result about $S_m(A)$ itself. If, on the other hand, A has at least one eigenvalue with multiplicity greater than one, several of the nice algebraic properties of Theorem 3.6 no longer hold. In particular, commutativity and closure do not hold in general for $S_m(A)$, and the cardinality of $S_m(A)$ will be infinite. The underlying reasons for the latter are that $\bigcup_X S_m(A, X) \subseteq S_2(A)$ from (2), that there are now infinitely many choices for the normalized columns of A 's eigenvector matrix X , and that $S_m(A, X_i)$ and $S_m(A, X_j)$ can differ when X_i and X_j differ. We will be more precise about this in what follows.

First, however, we prove a statement about the commonality between the sets $S_m(A, X_i)$ and $S_m(A, X_j)$. Since the identity matrix belongs in every set $S_m(A, X)$, they are clearly not disjoint. In fact, one can always find at least m elements in common between any pair of these sets, and when A does not have distinct eigenvalues a larger lower bound can be established. We begin by establishing a preliminary lemma.

LEMMA 4.1. *Let $X_i \in \mathbb{C}^{n \times n}$ and $X_j \in \mathbb{C}^{n \times n}$ be nonsingular matrices where the first $0 \leq r \leq n$ columns of X_j are a linear combination of the first r columns of X_i arising from right multiplication by a nonsingular linear transformation which leaves the remaining columns of X_i fixed. Let $D \in \mathbb{C}^{n \times n}$ be a diagonal matrix of the form*

$$D = \begin{pmatrix} \alpha I_r & 0 \\ 0 & D_{22} \end{pmatrix}$$

where I_r is the $r \times r$ identity matrix and α is a fixed complex number. Then $X_i D X_i^{-1} = X_j D X_j^{-1}$.

Proof. By hypothesis, $X_j = X_i M$ where

$$(7) \quad M = \begin{pmatrix} M_{11} & 0 \\ 0 & I_{n-r} \end{pmatrix},$$

with $M_{11} \in \mathbb{C}^{r \times r}$ nonsingular and where I_{n-r} is the $(n-r) \times (n-r)$ identity matrix.

Let

$$(8) \quad X_i = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \text{ and } X_i^{-1} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

under the same partitioning as D and M . If we write D_{11} in place of αI_r to allow us to express the forthcoming equalities more generally, then we see that $K = X_j D X_j^{-1} = X_i M D M^{-1} X_i^{-1}$ is comprised of the blocks

$$(9) \quad K_{11} = X_{11} M_{11} D_{11} M_{11}^{-1} Y_{11} + X_{12} D_{22} Y_{21}$$

$$(10) \quad K_{12} = X_{11} M_{11} D_{11} M_{11}^{-1} Y_{12} + X_{12} D_{22} Y_{22}$$

$$(11) \quad K_{21} = X_{21} M_{11} D_{11} M_{11}^{-1} Y_{11} + X_{22} D_{22} Y_{21}$$

$$(12) \quad K_{22} = X_{21} M_{11} D_{11} M_{11}^{-1} Y_{12} + X_{22} D_{22} Y_{22}$$

This is easily shown to equal $X_i D X_i^{-1}$. $\square$

THEOREM 4.2. *Let $A \in \mathbb{C}^{n \times n}$ be diagonalizable and let its eigenvalues of multiplicity greater than one comprise a set of values $V = \{v_k\}_{k=1}^s$ where $0 \leq s < n$ (V can be empty). Let $\mu_k > 1$ be the multiplicity of A 's eigenvalue with value v_k for $1 \leq k \leq s$. Then if X_i and X_j are each eigenvector matrices of A , we have that*

$$|S_m(A, X_i) \cap S_m(A, X_j)| \geq m^{n-r+s} \text{ where } r = \sum_{i=1}^s \mu_i.$$

Proof. Recall that the eigenvector matrices X_i and X_j of A are assumed to have normalized columns. So, aside from scalar multiples of magnitude one, the columns of X_i and X_j can only differ in the columns corresponding to the eigenvalues of multiplicity greater than one.

For convenience, we will assume that the diagonal elements of the eigenvalue matrix $\Lambda = \text{diag}(\lambda_k)_{1 \leq k \leq n}$ are arranged so that the distinct eigenvalues λ_i are positioned in the last $n - r$ diagonal elements and the eigenvalues of multiplicity greater than one are set up consecutively in the first r diagonal elements as

$$\Lambda_{jj} = \begin{cases} v_1 & \text{for } 1 \leq j \leq \mu_1 \\ v_2 & \text{for } 1 + \mu_1 \leq j \leq \mu_1 + \mu_2 \\ \vdots & \\ v_s & \text{for } 1 + \sum_{i=1}^{s-1} \mu_i \leq j \leq r \end{cases}$$

This establishes an ordering for the columns of the matrices X_i and X_j .

Consider $X_i D X_i^{-1} \in S_m(A, X_i)$ and $X_j D X_j^{-1} \in S_m(A, X_j)$ where D has the form

$$(13) \quad D = \begin{pmatrix} \zeta^k I_{11} & 0 \\ 0 & D_{22} \end{pmatrix},$$

$\zeta = e^{2\pi i/m}$, and k is an integer in the range $0 \leq k \leq m - 1$. Application of Lemma 4.1 shows that $X_i D X_i^{-1} = X_j D X_j^{-1}$. Since there are m choices for $\zeta^k I_{11}$ and m^{n-r} choices for D_{22} , we have m^{n-r+1} matrices D of the form (13) for which $X_i D X_i^{-1} = X_j D X_j^{-1}$.

We now consider the 2×2 block repartitioning of D where the new upper-left block is the upper-left $(r - \mu_s) \times (r - \mu_s)$ portion of the previous $r \times r$ upper-left block $\zeta^k I_{11}$. This new partitioning effectively means that the last μ_s elements of D_{11} in the previous partitioning have been transferred to the D_{22} block of the new partitioning.

Another application of Lemma 4.1 shows that if D has the form (13) under the new partitioning, then here too $X_i D X_i^{-1} = X_j D X_j^{-1}$. The number of choices for D under this partitioning that are distinct from that found in the previous partitioning is $(m-1)m^{n-r+1}$. So, the number of choices for D under the two partitionings is $m^{n-r+1} + (m-1)m^{n-r+1} = m^{n-r+2}$.

We can now proceed inductively, reducing the upper-left block's size by μ_j at each stage, for $j = s-1$ through $j = 1$. $\square$

COROLLARY 4.3. *Let X_i and X_j be any two eigenvector matrices of A .*

1. *In the case where all eigenvalues of A are the same (i.e., multiplicity n), $|S_m(A, X_i) \cap S_m(A, X_j)| \geq m$.*
2. *In the case where all eigenvalues of A are distinct, $S_m(A, X_i) = S_m(A, X_j)$.*

Proof. For the first assertion, apply Theorem 4.2 with $r = n$ and $s = 1$. The m elements found between any pair of $S_m(A, X_i)$ and $S_m(A, X_j)$ are precisely the matrices $\zeta^j I$ for $0 \leq j \leq m-1$.

For the second assertion, apply Theorem 4.2 with $r = 0$ and $s = 0$ together with equation (3). $\square$

A cursory study of the situation where A is a multiple of the identity matrix I_n for $n > 1$ is sufficient to generate examples where commutativity and closure fail for elements of $S_m(A)$ under matrix multiplication. It's also not hard to show that there are infinitely many m -involutions of every dimension greater than one, thereby showing that $|S_m(\alpha I_n)|$ is infinite for every $n > 1$. The latter statement, in fact, holds for $S_m(A)$ where A is any diagonalizable matrix in $\mathbb{C}^{n \times n}$ with an eigenvalue of multiplicity greater than one.

THEOREM 4.4. *Suppose $A \in \mathbb{C}^{n \times n}$ is diagonalizable and has at least one eigenvalue of multiplicity greater than one. Then the cardinality of the set $S_m(A)$ is non-denumerable.*

Proof. As in the proof of Theorem 4.2, we assume that the elements of the diagonal eigenvalue matrix $\Lambda = (\Lambda_{ij})_{1 \leq i, j \leq n}$ are arranged so that any eigenvalues of multiplicity greater than one are placed consecutively in the upper-left. Let us reuse the partitions (7) and (8) where we take the size of the upper-left block to be 2×2 . We will also use the diagonal matrix partition

$$D = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix}$$

where we take

$$D_{11} = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$$

with $\zeta = e^{2\pi i/m}$ and consider D_{22} fixed.

Let

$$M_{11} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

and define $\tilde{M} = M_{11} D_{11} M_{11}^{-1}$.

Our expression for $K \in S_m(A)$ is comprised of the block equations (9), (10), (11), and (12). Since the only degrees of freedom in these equations are the elements m_{11} , m_{12} , m_{21} , and m_{22} , we may confine our attention to the products

$$(14) \quad X_{11} \tilde{M} Y_{11}$$

$$(15) \quad X_{11}\tilde{M}Y_{12}$$

$$(16) \quad X_{21}\tilde{M}Y_{11}$$

$$(17) \quad X_{21}\tilde{M}Y_{12}.$$

Computing $\tilde{M} = M_{11}D_{11}M_{11}^{-1}$, we find

$$(18) \quad \tilde{M} = \mu \begin{pmatrix} m_{11}m_{22} - \zeta m_{12}m_{21} & (\zeta - 1)m_{11}m_{12} \\ (1 - \zeta)m_{21}m_{22} & -m_{12}m_{21} + \zeta m_{11}m_{22} \end{pmatrix}$$

where $\mu = (m_{11}m_{22} - m_{12}m_{21})^{-1}$. For convenience, assume the partitioning

$$\tilde{M} = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} \\ \tilde{m}_{21} & \tilde{m}_{22} \end{pmatrix}.$$

In (18), we note that the magnitude of the complex number $\tilde{m}_{12}$ ranges continuously over $[0, \infty)$ by (for example) setting $m_{22} = 0$, fixing m_{12} and m_{21} at nonzero values, and letting m_{11} vary. Similarly, we note that the magnitude of $\tilde{m}_{21}$ ranges continuously over $[0, \infty)$ by setting $m_{11} = 0$, fixing m_{12} and m_{21} at nonzero values, and letting m_{22} vary.

Let $z = m_{11}m_{22}$ and set $m_{12}m_{21} = 1$. Under this choice, $\tilde{m}_{11}$ becomes the complex function $\frac{z-\zeta}{z-1}$ and $\tilde{m}_{22}$ becomes the complex function $\frac{\zeta z-1}{z-1}$. From this, we see that here too, the magnitude of the elements $\tilde{m}_{11}$ and $\tilde{m}_{22}$ each range continuously over $[0, \infty)$.

From (8), we observe that the blocks X_{11} and X_{21} cannot be simultaneously zero. Similarly, Y_{11} and Y_{12} cannot be simultaneously zero. This leads us to consider four separate cases. Namely,

1. $X_{11} \neq 0$ and $Y_{11} \neq 0$
2. $X_{11} \neq 0$ and $Y_{12} \neq 0$
3. $X_{21} \neq 0$ and $Y_{11} \neq 0$
4. $X_{21} \neq 0$ and $Y_{12} \neq 0$

At least one of these cases must be true.

For Case 1, we focus on equation (14). Since $X_{11} \neq 0$, there exists at least one row of X_{11} that is nonzero. Pick one such row, and let it have values $x = \begin{pmatrix} \alpha & \beta \end{pmatrix}$. Since $Y_{11} \neq 0$, there exists at least one column of Y_{11} that is nonzero. Pick one such column and let it have values $y^T = \begin{pmatrix} \gamma & \delta \end{pmatrix}^T$. Then,

$$(19) \quad x\tilde{M}y = \gamma\alpha\tilde{m}_{11} + \gamma\beta\tilde{m}_{21} + \delta\alpha\tilde{m}_{12} + \delta\beta\tilde{m}_{22}$$

Since α and β are not simultaneously zero and γ and δ are not simultaneously zero, at least one of the four summands in (19) must not be identically nonzero. If only one summand is nonzero, since we know the magnitude of $\tilde{m}_{11}$, $\tilde{m}_{21}$, $\tilde{m}_{12}$, and $\tilde{m}_{22}$ each range continuously over $[0, \infty)$ for varying choices of m_{11} , m_{12} , m_{21} , and m_{22} , we have that $x\tilde{M}y$ assumes a non-denumerable set of values. So, suppose at least two of the four summands in (19) are not identically zero. We wish to show that $x\tilde{M}y$ cannot be limited to a discrete set of values. Equation (18) shows that $\tilde{m}_{11}$, $\tilde{m}_{21}$, $\tilde{m}_{12}$, and $\tilde{m}_{22}$ are continuous complex valued functions, and only $\tilde{m}_{11}$ and $\tilde{m}_{22}$ can be linearly related (e.g., when $\zeta = -1$). But if the coefficients of $\tilde{m}_{11}$ and $\tilde{m}_{22}$ are both nonzero in the linear combination (19), then the coefficients of $\tilde{m}_{21}$ and $\tilde{m}_{12}$

will also be nonzero. Therefore, if at least two of the four summands in (18) are nonzero, there will be a nonzero term involving either the product $\mu m_{11} m_{12}$ (i.e., $\tilde{m}_{12}$) or the product $\mu m_{21} m_{22}$ (i.e., $\tilde{m}_{21}$). Since only one of the four summands in (19) involves the product $\mu m_{11} m_{12}$ and only one of the four summands (19) involves the product $\mu m_{21} m_{22}$, the product $x\tilde{M}y$ cannot be limited to a discrete set of values. We conclude that the product in (14) assumes a non-denumerable number of values under the assumptions of Case 1.

The treatment of Cases 2, 3, and 4 are treated in the same manner, with Case 2 dealing with equation (15), Case 3 dealing with equation (16), and Case 4 dealing with equation (17). Since the form of D used in the demonstration occurs in every set $S_m(A, X)$ for which A has dimension two or more, and since at least one of the four cases must be true this completes the proof. $\square$

5. The Anti-Commuting Case. In addition to studying K_2 -symmetry, the papers [6], [7], and [10], also investigated matrices A that satisfied the anti-commuting relationship $AK = -KA$ where K is an involution (the K_2 -skew-centrosymmetric matrices). It would therefore be desirable to obtain results similar to those found for $S_m(A)$ that hold for the set of complex m -involutions that anti-commute with a fixed complex matrix $A \in \mathbb{C}^{n \times n}$.

Let us denote this set as $\tilde{S}_m(A)$ and suppose K is an m -involution. Some facts are immediately apparent. From a trivial determinant argument, we see that if n is odd and A is nonsingular, then $\tilde{S}_m(A)$ is empty. With just a little more effort, since $A = K^m A = (-1)^m AK^m = (-1)^m A$, if m is odd it then follows that $\tilde{S}_m(A)$ is empty except when A is the zero matrix, in which case $\tilde{S}_m(A)$ is non-denumerable for $n > 1$. So, we only need to focus on the case where m is even.

A deeper study of $\tilde{S}_m(A)$ requires a characterization of the structure of both A and K when $AK = -AK$. To accomplish this, we turn to the Jordan decomposition of A $\tilde{A} = S^{-1}AS$ of A , where

$$\tilde{A} = \text{diag}(\mathcal{J}_1(\lambda_1), \dots, \mathcal{J}_k(\lambda_k)),$$

the Jordan blocks $\mathcal{J}_i$ are $n_i \times n_i$, and $n = \sum n_i$. Let $\tilde{K}$ be any matrix of the same size as A , and block partition it so that its $\tilde{K}_{ij}$ block is $n_i \times n_j$. If we let $K = \tilde{K}S^{-1}$, then $AK = -KA$ if and only if $\tilde{A}\tilde{K} = -\tilde{K}\tilde{A}$ if and only if $\mathcal{J}_i\tilde{K}_{ij} = -\tilde{K}_{ij}\mathcal{J}_i$ for all i, j . The following lemma helps describe the structure of the blocks $\mathcal{J}_i$ and $\tilde{K}_{ij}$ whenever $\tilde{A}\tilde{K} = -\tilde{K}\tilde{A}$.

LEMMA 5.1. Suppose $B \in \mathbb{C}^{m \times n}$, I_m is the $m \times m$ identity matrix, and N_m is the $m \times m$ matrix with ones directly above the main diagonal and zeros elsewhere. Let $b_{i,j}$ denote the i, j element of the matrix B . Then

1. If $\lambda_i + \lambda_j \neq 0$, then

$$(20) \quad (\lambda_i I_m + N_m)B = -B(\lambda_j I_n + N_n)$$

if and only if $B = 0$.

2. If $\lambda_i + \lambda_j = 0$, then

$$(21) \quad (\lambda_i I_m + N_m)B = -B(\lambda_j I_n + N_n)$$

if and only B is upper triangular, where the components of columns 1 through $n - m$ are zero if $n > m$, the components of rows $m + 1$ through n are zero if $n < m$, and each diagonal of the upper triangle has constant magnitude and alternates in sign along the diagonal (i.e., $b_{i,j} = -b_{i+1,j+1}$).

Proof. To prove the first assertion, first rewrite equation (20) as

$$(22) \quad ((\lambda_i + \lambda_j)I_m + N_m)B + BN_n = 0.$$

Multiplying (22) on the right by N_n^j for $1 \leq j \leq n-1$ gives $n-1$ equalities of the form

$$(23) \quad ((\lambda_i + \lambda_j)I_m + N_m)BN_n^j + BN_n^{j+1} = 0.$$

Since $N_n^n = 0$ and $(\lambda_i + \lambda_j)I_m + N_m$ is nonsingular, if we let $j = n-1$ in equation (23), we have that $BN_n^{n-1} = 0$. Substituting this into equation (23) for $j = n-2$, we see that $BN_n^{n-2} = 0$. Continuing in this manner, we arrive at $BN_n = 0$ when $j = 1$ which, when substituted into (22), gives us $B = 0$.

To prove the second assertion, we first simplify equation (21) as

$$(24) \quad N_m B = -BN_n.$$

Comparison of the first column of each side of (24) shows that the matrix B is zero in the first column below the $b_{1,1}$ element. Equating the j -th column of each side of (24) gives

$$(25) \quad (b_{2,j}, b_{3,j}, \dots, b_{n,j}, 0)^T = -(b_{1,j-1}, b_{2,j-1}, \dots, b_{n-1,j-1}, b_{n,j-1})^T.$$

This shows that the diagonals of B are of equal magnitude and alternate in sign, and are zero below the main diagonal. Finally, we note that the m -th row of (24) shows that $b_{m,1} = b_{m,2} = \dots = b_{m,n-1} = 0$. Combining this with (25) shows that columns one through $n-m$ are zero when $n > m$. $\square$

The next lemma will be useful in the determination of the cardinality of $\tilde{S}_m(A)$.

LEMMA 5.2. *Suppose $B \in \mathbb{C}^{n \times n}$ has the form described in Lemma 5.1 and $n > 1$. That is, B is upper triangular and each element satisfies $b_{i,j} = -b_{i+1,j+1}$. Then, there are uncountably many involutory B .*

Proof. Let $C = B^2$. Then

$$c_{i,j} = \sum_{k=i}^j b_{i,k} b_{k,j} = \sum_{k=1}^{j-i+1} (-1)^{k+1} b_{1,k} b_{1,j-k-i+2}$$

Since $c_{i,j_1} = c_{i_2,j_2}$ whenever $j_2 - i_2 = j_1 - i_1$, we see that C is upper triangular Toeplitz and so we can restrict our attention to the values of the first row of C . For convenience, let us write c_j for $c_{1,j}$ and b_j for $b_{1,j}$. Then,

$$c_j = \sum_{k=1}^j (-1)^{k+1} b_k b_{j-k+1}.$$

When j is even, $c_j = 0$. When $j = 2m+1$, we have

$$(26) \quad c_{2m+1} = (-1)^{m+1} b_{m+1}^2 + 2 \sum_{k=1}^m (-1)^{k+1} b_k b_{2m-k+2}.$$

We want $C = I$. The constraint $c_1 = 1$ yields two solutions for b_1 . If $n = 2$, then b_2 is unconstrained and we are done. Assume $n > 2$. Since the solutions for b_1

can be substituted back into (26), the equation $c_3 = 0$ becomes a complex bivariate quadratic in b_2 and b_3 which has uncountably many solutions for b_2 and b_3 . This process can be continued as m is incremented. Each time m is increased by one, the solutions of the previous iterations can be substituted into (26) and two new unknowns b_{2m} and b_{2m+1} appear, resulting in another bivariate quadratic for which there are uncountably many solutions. $\square$

THEOREM 5.3. *Given $A \in \mathbb{C}^{n \times n}$ and an m -involution $K \in \mathbb{C}^{n \times n}$:*

1. *Let $n = 1$. If A is the number zero, then $\tilde{S}_m(A)$ contains the m -th roots of unity. Otherwise, $\tilde{S}_m(A)$ is empty.*
2. *Suppose $n > 1$ and let $m > 1$ be an odd integer. If A is the zero matrix, then $\tilde{S}_m(A)$ is non-denumerable. Otherwise, $\tilde{S}_m(A)$ is empty.*
3. *Suppose $n > 1$ and let $m > 1$ be an even integer. If the non-zero eigenvalues of A come in pairs of opposite sign where the corresponding pairs have Jordan blocks of equal size, then the cardinality of the set $\tilde{S}_m(A)$ is non-denumerable. Otherwise $\tilde{S}_m(A)$ is empty.*

Proof. The first assertion is trivial, while the second assertion's proof was given at the beginning of this section. We proceed with the proof of the third assertion.

From Lemma 5.1, we see that if A anti-commutes with K then its nonzero eigenvalues must come in pairs of opposite sign where the corresponding pairs have Jordan blocks of equal size. So, to demonstrate the theorem's third assertion we only need to show that $\tilde{S}_m(A)$ is non-denumerable when A satisfies this condition. For convenience, we will assume that the pairs of Jordan blocks with eigenvalues of opposite sign appear consecutively along the diagonal of $\tilde{A}$. The proof will be broken into several cases.

1. Case: $\tilde{A}$ is the zero matrix.
2. Case: $\tilde{A}$ has all eigenvalues zero and has at least Jordan block that is $m \times m$ where $m > 1$.
3. Case: $\tilde{A}$ has at least one pair of Jordan blocks for non-zero eigenvalues of opposite sign that are both $m \times m$ where $m > 1$.
4. Case: All Jordan blocks of $\tilde{A}$ corresponding to non-zero eigenvalues are 1×1 .

Case 1 is trivial, so we move on to Case 2. For Case 2, we consider a particular class of block-diagonal matrices $\tilde{K}$ whose diagonal blocks are the same size as those of $\tilde{A}$. Assume without loss of generality that an $m \times m$ Jordan block of $\tilde{A}$, where $m > 1$, occupies the 1,1 block position. Consider the set of block matrices whose 1,1 block is an upper triangular involution whose elements satisfy $k_{i,j} = -k_{i+1,j+1}$, and whose remaining diagonal blocks are diagonal matrices with alternating values of 1 and -1 . The elements of this set are clearly involutions, they anti-commute with $\tilde{A}$ since their form satisfies the conditions of Lemma 5.1, and Lemma 5.2 shows that there are uncountably many choices for the 1,1 block. Since m is even, this set of involutions belongs to $\tilde{S}_m(A)$ and so we are done with this case.

Case 3. Assume without loss of generality that the 1,1 and 2,2 Jordan block positions of $\tilde{A}$ are occupied by blocks corresponding to a positive-negative eigenvalue pair and whose block sizes are $m \times m$ where $m > 1$. We choose the 1,2 and 2,1 blocks of $\tilde{K}$ such that $\tilde{K}_{1,2} = \tilde{K}_{2,1}$ and $\tilde{K}_{1,2}$ and $\tilde{K}_{2,1}$ are upper triangular involutions whose elements satisfy $k_{i,j} = -k_{i+1,j+1}$. For the remaining $\tilde{A}$ Jordan block pairs in the i,i and $i+1,i+1$ positions, pick the $\tilde{K}$ blocks in the $i,i+1$ and $i+1,i$ positions to be identical diagonal matrices with alternating 1's and -1 's on their diagonals. For the $\tilde{A}$ blocks in the i,i positions corresponding to zero eigenvalues, let the i,i block of $\tilde{K}$ also be diagonal with alternating 1's and -1 's. Assume the remaining blocks of $\tilde{K}$ are

zero. Then $\tilde{K}$ is an involution, Lemma 5.2 shows that there are uncountably many choices for $\tilde{K}$'s $1, 1$ and $2, 2$ blocks, and Lemma 5.1 shows that $\tilde{K}$ anti-commutes with $\tilde{A}$. As noted in Case 2, these involutions belong to $\tilde{S}_m(A)$.

Case 4. Assume without loss of generality that the $1, 1$ and $2, 2$ Jordan block positions of $\tilde{A}$ are occupied by 1×1 blocks corresponding to a positive-negative eigenvalue pair. Pick elements $k_{1,2}$ and $k_{2,1}$ of $\tilde{K}$ such that $k_{1,2}k_{2,1} = 1$. For the remaining 1×1 Jordan block pairs of $\tilde{A}$ in the i, i and $i+1, i+1$ positions, pick the elements $k_{i,i+1}$ and $k_{i+1,i}$ to be 1. For the $\tilde{A}$ blocks in the i, i positions corresponding to zero eigenvalues, let the i, i block of $\tilde{K}$ be diagonal with alternating 1's and -1 's (as in Case 3) and assume the remaining blocks of $\tilde{K}$ are zero. Then $\tilde{K}$ is an involution, there are uncountably many choices for $k_{1,2}$ and $k_{2,1}$, and Lemma 5.1 shows that $\tilde{K}$ anti-commutes with $\tilde{A}$. As noted in Case 2, these involutions belong to $\tilde{S}_m(A)$.

Since these results for $\tilde{A}$ and $\tilde{K}$ in Cases 1-4 translate directly into anti-commuting results for A and K , we are done. $\square$

Lemma 5.1, together with the m -involution constraint, characterize the matrices $\tilde{K}$ which in turn characterizes the elements of $\tilde{S}_m(A)$. This characterization can then be used to develop an algorithm for generating subsets of $\tilde{S}_m(A)$. For example, after computing the Jordan decomposition $\tilde{A}$ of A , one could construct examples of $\tilde{K}$ as follows. For the $\tilde{A}$ Jordan block pairs corresponding to non-zero eigenvalues in the i, i and $i+1, i+1$ block positions, pick the $\tilde{K}$ blocks in the $i, i+1$ and $i+1, i$ positions to be identical diagonal matrices whose $1, 1$ diagonal element is an m -th root of unity and whose subsequent diagonal elements alternate in sign. For the $\tilde{A}$ blocks in the i, i positions corresponding to zero eigenvalues, let the i, i block of $\tilde{K}$ also be diagonal whose $1, 1$ diagonal element is an m -th root of unity and whose subsequent diagonal elements alternate in sign. Assume the remaining blocks of $\tilde{K}$ are zero.

Of course, there are many other elements of $\tilde{S}_m(A)$ that this particular construction of $\tilde{K}$ does not consider. For example, other blocks of $\tilde{K}$ may be nonzero, in particular if there multiple Jordan blocks of $\tilde{A}$ corresponding to a particular eigenvalue value. Also, it's not necessary for the blocks of $\tilde{K}$ to be diagonal. The example construction described in the previous paragraph could be modified to include such possibilities. The main drawback to this construction, however, is its reliance on the Jordan decomposition, which is known to be very sensitive numerically. Constructing an efficient and stable algorithm for generating elements of $\tilde{S}_m(A)$ that does not rely on first computing the Jordan decomposition of A is a possible topic for future investigation.

6. Concluding Remarks. This paper has explored some of the fundamental properties of the class $S_m(A)$ of m -involutory matrices that commute with a given diagonalizable matrix A . The constructive nature of the definition of $S_m(A, X)$ allows one to easily generate numerous (in some cases all) m -involutions commuting with A . By providing a constructive means to generate elements of $S_m(A)$, it now becomes easier to identify the types K_m -symmetry satisfied by a matrix A and thereby use the body of results that has accumulated in recent years regarding K_m -symmetry. Other results were given for the class $\tilde{S}_m(A)$ of m -involutory matrices that anti-commute with A . It is hoped that the results obtained in this paper will lead to additional insights and research related to the class $S_m(A)$, $\tilde{S}_m(A)$, the study of K_m -symmetric matrices, and their applications.

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Appendix A. An Open Question (Unpublished Addendum). Let $A \in \mathbb{C}^{n \times n}$ be diagonalizable with an eigenvector matrix X . Theorem 4.4 shows that when the matrix A has at least one eigenvalue of multiplicity greater than one that there exists a non-denumerable number of distinct sets $S_m(A, X)$ as X varies over all possible eigenvector matrices of A . While the sets $S_m(A, X)$ clearly have non-trivial overlap (and Theorem 4.2 gives a lower bound on the cardinality of the intersection), no attempt was made earlier to characterize the interaction between *unshared* elements of distinct $S_m(A, X_i)$ and $S_m(A, X_j)$.

The following is a basic open conjecture regarding this interaction:

CONJECTURE A.1. *Let $A \in \mathbb{C}^{n \times n}$ be diagonalizable and suppose $K_i \in S_m(A, X_i)$ and $K_j \in S_m(A, X_j)$, where $K_i \notin S_m(A, X_i) \cap S_m(A, X_j)$ and $K_j \notin S_m(A, X_i) \cap S_m(A, X_j)$. Then there does not exist an eigenvector matrix X_k of A where either $K_i K_j \in S_m(A, X_k)$ or $K_j K_i \in S_m(A, X_k)$ is satisfied, and $S_m(A, X_k) \neq S_m(A, X_i)$ and $S_m(A, X_k) \neq S_m(A, X_j)$.*

Appendix B. The Jordan Block Structure of Anti-Commuting Matrices (Unpublished Addendum). Assume that the matrix $A \in \mathbb{C}^{n \times n}$ is not the zero matrix. When $\tilde{S}_m(A)$ is non-empty, part three of Theorem 5.3 asserts that any nonzero eigenvalues of A must appear in positive and negative pairs and that the Jordan blocks associated with each pairing must have identical block size. A proof of this simple fact, which was omitted earlier, is provided in this appendix.

The statement that we'll prove, which is somewhat more general than what is needed for Theorem 5.3, is given in the following lemma.

LEMMA B.1. *Let $K \in \mathbb{C}^{n \times n}$ be nonsingular, and suppose $AK = -KA$. Then the nonzero eigenvalues of A must appear in positive-negative pairs, and each Jordan block for an eigenvalue $\lambda \neq 0$ must have a corresponding Jordan block for $-\lambda$ of identical size.*

Proof. Given a complex number λ , let $G_\lambda = \{x \in \mathbb{C}^n \mid (A - \lambda I)^k x = 0 \text{ for some } k > 0\}$ be the generalized eigenspace corresponding to λ . For any $x \in G_\lambda$ and $k > 0$ such that $(A - \lambda I)^k x = 0$, we have

$$(27) \quad 0 = K(A - \lambda I)^k x = (-A - \lambda I)^k Kx = (-1)^k (A + \lambda I)^k Kx.$$

Therefore, $Kx \in G_{-\lambda}$ and so $K(G_\lambda) \subseteq G_{-\lambda}$.

If λ is an eigenvalue of A , then its multiplicity $m(\lambda)$ equals the dimension, $d(G_\lambda)$, of G_λ . Therefore,

$$m(\lambda) = d(G_\lambda) = d(KG_\lambda) \leq d(G_{-\lambda}) = m(-\lambda)$$

where the second equality follows from the nonsingularity of K . Since $-\lambda$ is an eigenvalue of A by (27) with $k = 1$, we can apply the same argument to $-\lambda$ to get that $m(-\lambda) \leq m(\lambda)$. Therefore the multiplicity of λ and $-\lambda$ is the same.

Now let $B_\lambda = \{x, (A - \lambda I)x, \dots, (A - \lambda I)^{k-1}x\}$ be a cyclical basis for a k -dimensional Jordan block of A for the eigenvalue λ . In other words, $(A - \lambda I)^{k-1}x \neq 0$, $(A - \lambda I)^k x = 0$, and x is not in the range of $(A - \lambda I)$.

We wish to show that $B_{-\lambda} = \{Kx, (A + \lambda I)Kx, \dots, (A + \lambda I)^{k-1}Kx\}$ is a cyclical basis for a k -dimensional Jordan block of A corresponding to $-\lambda$, where x is the

same vector used to construct B_λ . First, note that

$$(A + \lambda I)^{k-1} Kx = (-1)^{k-1} K(A - \lambda I)^{k-1} x \neq 0$$

since K is non-singular. Similarly,

$$(A + \lambda I)^k Kx = (-1)^k K(A - \lambda I)^k x = 0.$$

Now suppose that $Kx = (A + \lambda I)y$ for some y . Then,

$$Kx = (A + \lambda I)KK^{-1}y = -K(A - \lambda I)K^{-1}y.$$

In other words, $x = -(A - \lambda I)K^{-1}y$, which contradicts the fact that x is not in the range of $(A - \lambda I)$. Therefore, Kx is not in the range of $(A + \lambda I)$, and we conclude that $B_{-\lambda}$ is in fact a cyclical basis for a Jordan block of the same size as B_λ .

Clearly, $m(\lambda)$ is equal to the sum of the size of all Jordan blocks for the eigenvalue λ , and $m(-\lambda)$ is equal to the sum of the size of all Jordan blocks for the eigenvalue $-\lambda$. Since we've already established that $m(\lambda) = m(-\lambda)$, the lemma is proved. $\square$

Appendix C. Some Simple Examples Illustrating the Construction of Elements in $\tilde{S}_m(A)$ (Unpublished Addendum). As before, we let $\tilde{S}_m(A)$ denote the class of m -involutions that anti-commute with a fixed matrix $A \in \mathbb{C}^{n \times n}$. In what follows, assume that the matrix A is not the zero matrix.

Although not explicitly spelled out earlier, Lemma 5.1 provides a means by which elements of $\tilde{S}_m(A)$ can be constructed for $n > 1$ and even $m \geq 2$ (as noted in Theorem 5.3, $\tilde{S}_m(A)$ is empty when both A is nonzero and m is odd are simultaneously true). The proof of Theorem 5.3 and some of its subsequent discussion are suggestive of how this construction can be performed. In this appendix, we provide some simple examples to make this more concrete for the interested reader.

Let $\tilde{A} = X^{-1}AX$ denote the Jordan canonical form of A . If $\tilde{A}\tilde{K} = -\tilde{K}\tilde{A}$ for some m -involution $\tilde{K}$, then it's clear that $K = X\tilde{K}X^{-1}$ is an m -involution satisfying $AK = -KA$. Consequently, focusing our attention to the Jordan canonical form of A in the following examples is by no means restrictive.

C.1. Diagonalizable Matrices with Two Distinct Nonzero Eigenvalues.

We begin with an easy exercise, treating the general case of a diagonalizable matrix $\tilde{A} \in \mathbb{C}^{n \times n}$ in Jordan canonical form, where $\tilde{A}$ has exactly two distinct eigenvalues, neither of which are zero. We will assume that $\tilde{S}_m(\tilde{A})$ is non-empty.

First, we note that from Theorem 5.3 that if $\tilde{S}_m(\tilde{A})$ is non-empty, then $\tilde{A}$ must have each of its nonzero eigenvalues appear in positive-negative pairs. For convenience, we write the the Jordan canonical form of $\tilde{A}$ with the following arrangement of its eigenvalues:

$$\begin{pmatrix} \lambda & & & & & \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & -\lambda & & \\ & & & & \ddots & \\ & & & & & -\lambda \end{pmatrix}$$

In this example, Lemma 5.1's elements $b_{i,j}$, which correspond to the elements of the prospective m -involution $\tilde{K}$, must be zero in the upper left and lower right

$n/2 \times n/2$ matrix blocks since $\lambda_i + \lambda_j = 2\lambda_i = 2\lambda_j \neq 0$ for these indices. Part 2 of Lemma 5.1 does not impose any further restrictions on the form of $\tilde{K}$, and so we have the block representation:

$$\tilde{K} = \begin{pmatrix} 0 & K_{12} \\ K_{21} & 0 \end{pmatrix}$$

$\tilde{A}$ and $\tilde{K}$ clearly satisfy the relation $\tilde{A}\tilde{K} = -\tilde{K}\tilde{A}$. The condition that $\tilde{K}$ be an m -involution is not specifically addressed by Lemma 5.1 (the lemma only establishes anti-commuting constraints), but is clearly satisfied by imposing that $(K_{12}K_{21})^r = I_{n/2 \times n/2}$, where $I_{n/2 \times n/2}$ is the $n/2 \times n/2$ identity matrix and $m = 2r$ (as noted earlier, we only need to consider the cases where m is even). In the special case where $\lambda = 1$ and $m = 2$, we note that this example reproduces an observation made by Kestelman in section 3.i of [5].

C.2. Defective Matrices with Two Jordan Blocks and Nonzero Eigenvalues. We next look at the case of a defective matrix $\tilde{A} \in \mathbb{C}^{n \times n}$ in Jordan canonical form with two Jordan blocks corresponding to nonzero eigenvalues. Since $\tilde{S}_m(\tilde{A})$ will assumed to be non-empty, we note again that Theorem 5.3 states that $\tilde{A}$'s two eigenvalues will appear as positive negative pairs λ and $-\lambda$ and that the Jordan blocks will have equal size. That is, $\tilde{A}$ can be written in the form

$$\begin{pmatrix} \lambda & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & 1 & & & & \\ & & & \lambda & 0 & & & \\ & & & & -\lambda & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & & & & -\lambda \end{pmatrix}$$

Let $\tilde{K}$ have a compatible block-partitioning with $\tilde{A}$'s Jordan blocks, which we label as

$$\tilde{K} = \begin{pmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ \tilde{K}_{21} & \tilde{K}_{22} \end{pmatrix}.$$

Part 1 of Lemma 5.1 shows that $\tilde{K}_{11}$ and $\tilde{K}_{22}$ must be zero matrix blocks. Part 2 of Lemma 5.1 shows that the block $\tilde{K}_{12}$ has the form

$$\begin{pmatrix} k_1 & k_2 & k_3 & \cdots & \cdots & \cdots & k_{n-1} & k_n \\ & -k_1 & -k_2 & -k_3 & & & & -k_{n-1} \\ & & k_1 & k_2 & k_3 & & & \vdots \\ & & & -k_1 & -k_2 & -k_3 & & \vdots \\ & & & & \ddots & \ddots & \ddots & \vdots \end{pmatrix}$$

Similarly for $\tilde{K}_{21}$.

The m -involution constraint then forces

$$\left(\tilde{K}_{12}\tilde{K}_{21}\right)^r = \left(\tilde{K}_{21}\tilde{K}_{22}\right)^r = I_{n/2 \times n/2}$$

where $m = 2r$ and $I_{n/2 \times n/2}$ is the $n/2 \times n/2$ identity matrix.

In the special case where $\tilde{A} \in C^{4 \times 4}$, $\tilde{K}$'s blocks can be written as

$$\tilde{K}_{12} = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}.$$

and the block $\tilde{K}_{21}$ can be written as

$$\tilde{K}_{21} = \begin{pmatrix} c & d \\ 0 & -c \end{pmatrix}.$$

The m -involution constraint can then be expressed as $(ac)^r = 1$ and $ad - bc = 0$.

C.3. One More Example. We finish this appendix with a slightly more complicated but concrete example, with less discussion now that the application of Lemma 5.1 is familiar.

Given $\lambda \neq 0$, let $\tilde{A} \in \mathbb{C}^{12 \times 12}$ have the Jordan canonical form

$$\tilde{A} = \begin{pmatrix} J_1 & & & & \\ & J_2 & & & \\ & & J_3 & & \\ & & & J_4 & \\ & & & & J_5 \end{pmatrix}$$

where

$$J_1 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

$$J_2 = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{pmatrix},$$

$$J_3 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

$$J_4 = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix},$$

and

$$J_5 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This form for $\tilde{A}$ is compatible with $\tilde{S}_m(\tilde{A})$ being non-empty.

Then $\tilde{K}$ will have the form

$$\tilde{K} = \begin{pmatrix} & K_{12} & & K_{14} & \\ K_{21} & & K_{23} & & \\ & K_{32} & & K_{34} & \\ K_{41} & & K_{43} & & \\ & & & & K_{55} \end{pmatrix}$$

where

$$K_{12} = \begin{pmatrix} a_{12} & b_{12} & c_{12} \\ 0 & -a_{12} & -b_{12} \\ 0 & 0 & a_{12} \end{pmatrix},$$

$$K_{21} = \begin{pmatrix} a_{21} & b_{21} & c_{21} \\ 0 & -a_{21} & -b_{21} \\ 0 & 0 & a_{21} \end{pmatrix},$$

$$K_{14} = \begin{pmatrix} a_{14} & b_{14} \\ 0 & -a_{14} \\ 0 & 0 \end{pmatrix},$$

$$K_{23} = \begin{pmatrix} a_{23} & b_{23} \\ 0 & -a_{23} \\ 0 & 0 \end{pmatrix}$$

$$K_{32} = \begin{pmatrix} 0 & a_{32} & b_{32} \\ 0 & 0 & -a_{32} \end{pmatrix},$$

$$K_{41} = \begin{pmatrix} 0 & a_{41} & b_{41} \\ 0 & 0 & -a_{41} \end{pmatrix},$$

$$K_{34} = \begin{pmatrix} a_{34} & b_{34} \\ 0 & -a_{34} \end{pmatrix},$$

$$K_{43} = \begin{pmatrix} a_{43} & b_{43} \\ 0 & -a_{43} \end{pmatrix},$$

and

$$K_{55} = \begin{pmatrix} a_{55} & b_{55} \\ 0 & -a_{55} \end{pmatrix}.$$

Constraints on the values of a_{ij} , b_{ij} , and c_{ij} are imposed by the m -involution constraint on $\tilde{K}$. These constraints can be determined taking the m -th power of $\tilde{K}$ and setting the result equal to the identity matrix.

Appendix D. A MATLAB script For Generating m -Involutions (Unpublished Addendum). This appendix contains a basic MATLAB script for generating a particular subset of the class $S_m(A)$. When the matrix A is non-defective, it does this by computing the set $S_m(A, X)$ for some eigenvector matrix X of A , less those matrices that are simply m -th root of unity multiples of another element of $S_m(A, X)$.

```

function[kList] = kMatrices(A, m)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% K-matrix Generator
% Version 1.0
% Copyright (c) 2011 Mark Yasuda
%
% Revision 1.1: Introduced an additional dimension to the kMatrices
%               output to avoid concatenation of the list of
%               commuting m-involutions (07/31/2015).
%
% Permission is hereby granted, free of charge, to any person
% obtaining a copy of this software to use, copy, modify, merge,
% publish, and distribute it subject to the following request:
%
% That reference be made to the following work:
%
% [1] M. Yasuda, Some properties of commuting and anti-commuting
%      m-involutions, Acta Mathematica Scientia, Volume 32, No. 2
%      (2012), pp 631–644.
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% Sample usage:
% >> kList = kMatrices(A, m)
%
% Details: This function constructs commuting K-matrix m-involutions —
%          that is, given a square matrix A of dimension n, a list of
%          m-involutory matrices (i.e.,  $K^m = I$ ) satisfying  $AK = KA$  is
%          returned.
%
%          When the matrix A is non-defective, it does this by computing
%          the set  $S_m(A, X)$  for some eigenvector matrix X of A, less
%          those matrices K that are simply m-th roots of unity multiples
%          of another element of  $S_m(A, X)$ . See Definition 3.1 of the
%          paper [1] for the definition of  $S_m(A, X)$ .
%
% Inputs: Square matrix A (complex) and an integer  $m > 0$ 
%
% Outputs: Given valid inputs, a list of m-involutions K that commute
%          with A is returned.
%
%          Let A be a square matrix of dimension n. If A is
%          non-defective, then a total of  $m^{n-1}$  matrices K satisfying
%           $AK = KA$  are returned in a 3-dimensional array. Otherwise,
%          only the identity is returned.
%
% Example: A = [1 2 3; 4 5 6; 7 8 9];
%            kList = kMatrices(A, 2)
%            k1 = kList(:, :, 1)
%            k2 = kList(:, :, 2)
%            k3 = kList(:, :, 3)
%            k4 = kList(:, :, 4)
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% Comments:
%
% This m-file script is a quick hack. There are probably much better
% ways to write the code. Feel free to contact me with suggested
% improvements at:
%
% mark.yasuda@gmail.com
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

n = size(A, 1);
if (size(A(1)) ~= size(A(2)))
    disp('Input matrix is not square — returning the zero matrix');
    kList = zeros(n);
    return;
end

```

```

if (m < 1)
    disp('Second parameter should be an integer greater than zero:');
    disp('Returning the zero matrix');
    kList = zeros(n);
    return;
end

if (m == 1)
    disp('m = 1: Returning the identity matrix');
    kList = eye(n);
    return;
end

%%%% Eigendecomposition %%%%
[X, D] = eig(A);

D = eye(n);
sizeX = size(X);
if (size(X(1)) ~= size(X(2)))
    disp('Input matrix is defective — infinitely many solutions:');
    disp('Returning only the identity');
    kList = eye(n);
    return;
end

if (m == 2)
    % If there are distinct eigenvalues, there will be 2^n distinct
    % K matrices. We return half of them (no need to return the
    % additive inverses of the ones provided).
    kList = zeros(n, n, 2^(n-1));

    % Fix the last diagonal element of D at 1 to avoid returning
    % additive inverses of other K.
    % For m == 2, we exploit the binary representation of the index
    % to iterate through the 2^(n-1) elements.
    for index = 1:2^(n-1)
        D = eye(n);
        for j = 1:n-1
            % for the n-1 digits we are considering, assign
            % D's jth diagonal element a -1 if bit j is 1
            if (bitget(uint8(index), j) == 1)
                D(j,j) = -1;
            end
        end
        % D
        kList(:, :, index) = X*D*inv(X);
    end
else
    % If there are distinct eigenvalues, there will be m^n
    % distinct K matrices. We return a set that avoids
    % trivial multiples of m-th roots of unity of other
    % K-matrices.
    kList = zeros(n, n, m^(n-1));

    rootOfUnity = exp(2*pi*i/m);

    % Fix the last diagonal element of D at 1 to avoid
    % returning additive inverses of other K
    for iterationValue = 0:m^(n-1) - 1
        D = eye(n);

        % Note that one can utilize a base-m expansion to
        % iterate through the m^n solutions, m^(n-1) of
        % which are returned (avoiding scalar multiples of
        % other solutions). The base-m expansion uses the
        % MATLAB/Octave functions dec2base, substr, and
        % str2num to extract the base-m digits.
        baseStringExpansion = dec2base(iterationValue, m, n);
        for j = 2:n
            digitValue = str2num(substr(baseStringExpansion, j, 1));
            D(j,j) = power(rootOfUnity, digitValue);
        end
        indexValue = iterationValue + 1;
    end
end

```

```

        kList(:, :, indexValue) = X*D*inv(X);
    end
end

```

REFERENCES

- [1] I. T. ABU-JEIB, [Involutions and Generalized Centrosymmetric and Skew-Centrosymmetric Matrices](#), Canadian Applied Mathematics Quarterly, 15 (2007).
- [2] A. ANDREW, [Eigenvectors of certain matrices](#), Linear Algebra Appl., 7 (1973), pp. 151–162.
- [3] A. CANTONI AND P. BUTLER, [Eigenvalues and eigenvectors of symmetric centrosymmetric matrices](#), Linear Algebra Appl., 13 (1976), pp. 275–288.
- [4] A. COLLAR, [On centrosymmetric and centroskew matrices](#), Quart. J. Mech. Appl. Math., 15 (1962), pp. 265–281.
- [5] H. KESTELMAN, [Anticommuting linear transformations](#), Canadian Journal of Mathematics, Volume 13, No. 4 (1961), pp. 614–624
- [6] D. TAO AND M. YASUDA, [A Spectral Characterization of Generalized Real Symmetric Centrosymmetric and Generalized Real Symmetric Skew-Centrosymmetric Matrices](#), SIAM J. Matrix Anal. Appl., 23 (2002), pp. 885–895
- [7] W. F. TRENCH, [Characterization and properties of matrices with generalized symmetry or skew-symmetry](#), Linear Algebra Appl., 377 (2004), pp. 207–218
- [8] W. F. TRENCH, [Characterization and properties of matrices with \$k\$ -involutory symmetries](#), Linear Algebra Appl., 429 (2008), pp. 2278–2290
- [9] W. F. TRENCH, [Characterization and properties of matrices with \$k\$ -involutory symmetries](#), Linear Algebra Appl., In Press
- [10] M. YASUDA, [A Spectral Characterization of Hermitian Centrosymmetric and Hermitian Skew-Centrosymmetric \$K\$ -Matrices](#), SIAM J. Matrix Anal. Appl., 25 (2003), pp. 601–605