Chapter 4

Determinants of Graphs of Graphs

This chapter is divided into 6 sections. In section 4.1, we study the determinant of union of graphs with a common vertex. In section 4.2 , 4.3, 4.4, 4.5 and 4.6 we provide result for the determinant of $P_n(P_m)$, $C_n(P_m)$, $P_n(C_m)$, $W_n(m)$ and $C_n(C_m)$, respectively.

4.1 Union of Graphs with a common vertex

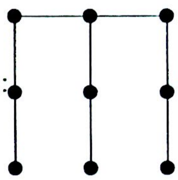
Definition 4.1.1. Suppose that graphs G and H are given. A graph F is said to be a $G - graph$ of $H's$ (or G of $H's$, in short) provided that

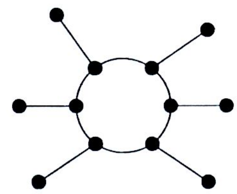
(1) $V(F) = V(G) \times V(H)$;

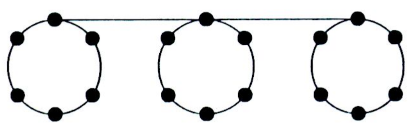
(2) $\{(u, x), (u, y)\} \in E(F) \Leftrightarrow \{x, y\} \in E(H)$, for each $u \in V(G)$ and $x, y \in V(H)$;

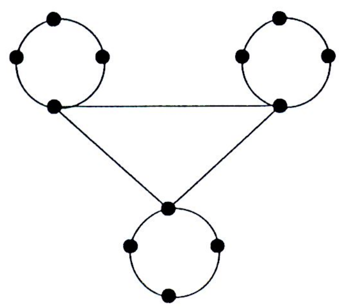
(3) for each $\{u, v\} \in E(G)$ there exists exactly one edge in F that joins (u, x) and (v, y), for some $x, y \in V(H)$, and if there is an edge in F that joins (u, x) and (v, y), with $u \neq v$, then $\{u, v\} \in E(G)$.

Note that the subgraph of F induced on the set $\{u\} \times V(H)$ is isomorphic to H, for all $u \in V(G)$. So, roughly speaking, one receives any G -graph of $H's$ if one substitutes each vertex of G by a copy of the graph H . One can see the operation of forming G-graphs of $H's$ as a (non-commutative) graph product and one easily shows its associativity. In the sequel, we will assume that $G's$ and $H's$ are paths and / or cycles and, in result, we will consider *paths of paths*, *cycles of paths*, *paths of cycles* and *cycles of cycles*.

$P_3(P_3)$: 

$C_6(P_2)$: 

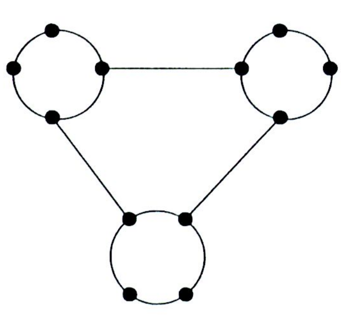
$P_3(C_6)$: 

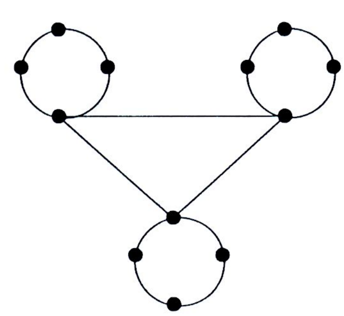
$C_3(C_4)$: 

To make the definition of paths(or cycles) of cycles unique one should reformulate the clause 3 of the above definition. So, let us define $G(C_m)$ as the G-graph of $C_m's$ such that

3') there is an edge in F that joins (u, x) with (v, y) $(for\ u \neq v)$ if and only if $\{u, v\} \in E(G)$ and $x = y = 1$.

Let us notice that $G(C_m)$ is defined uniquely. Observe that both graphs in the diagram above are 3-element paths of $C_6's$ but only the second of them is $P_3(C_6)$. Similarly, both graphs presented below are 3 - element cycles of $C_4's$ but only one of them is $C_3(C_4)$. We shall refer to graphs of the form $C_n(C_m)$, for $n, m \geq 3$, as *necklace* graphs.





It is clear that the adjacency matrix depends on the labeling of the vertices. However, we are interested in those properties of the matrix which are invariant under permutations of its row and columns corresponding to permutations of the sequence $v_1, v_2, ..., v_n$. Foremost among such properties is the value of the determinant of $A(G)$. In particular, we shall say that a graph G is *singular* if $\det A(G) = 0$. It is an immediate corollary of the following obvious (and well-known) lemma that a graph is singular if (at least) one of its components is singular.

Lemma 4.1.2. *If G and H are (vertex) disjoint graphs, then*

$$\det A(G \cup H) = \det A(G) \cdot \det A(H)$$

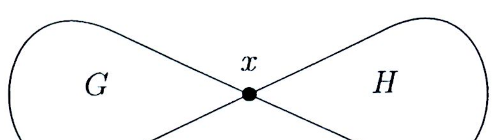
There are results in literature by use of which one can compute the determinant of the adjacency matrix of some graphs. In particular, see Proposition 2.3.8.

However, computations of the determinant of any $n \times n$ matrix for a large number n are generally difficult. One of the methods which applies there is the reduction of (computations of the determinant of the adjacency matrix for) certain graphs to its proper subgraphs. Rara's paper [22] contains some useful reduction formulas. Another result of this type is presented below.

Theorem 4.1.3. *Let G and H be graphs such that $V(G) \cap V(H) = \{x\}$.Then*

$$\det A(G \cup H) = \det A(G \backslash x) \cdot \det A(H) + \det A(H \backslash x) \cdot \det A(G).$$

$G \cup H$:



Proof. Let Γ be a sesquivalent spanning subgraph of $G \cup H$. Since $x \in V(\Gamma)$, there is a vertex $y \in V(\Gamma) = V(G) \cup V(H)$ such that $\{x, y\} \in E(\Gamma)$. We have the following two possibilities to consider.

1) $y \in V(G)$. If $\{x, y\}$ is a separate edge in Γ, then $\{x, z\} \in E(\Gamma)$ for none $z \in V(H)$. Similarly, if x and y are elements of a cycle in Γ, then the cycle must be included in G (as according to our assumptions, x is the only common vertex of the graph G and H), and hence $\{x, z\} \in E(\Gamma)$ for none $z \in V(H)$. In result, Γ is a subgraph of $G \cup (H \backslash x)$.

2) $y \in V(H)$. Then by an analogous argument, Γ is a subgraph of $(G \setminus x) \cup H$. Thus, each sesquivalent spanning subgraph of $G \cup H$ is, either a subgraph of $G \cup (H \backslash x)$, or of $(G \setminus x) \cup H$. Moreover, it cannot be a subgraph of both. By using Proposition 2.3.8 and Lemma 4.1.2, so

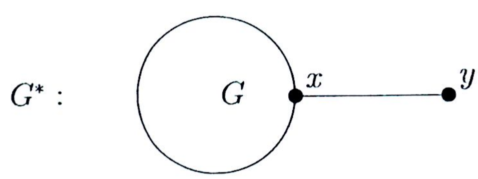
$$\det A(G \cup H) = \det A((G \setminus x) \cup H) + \det A((H \setminus x) \cup G)$$

$$= \det A(G \setminus x) \cdot \det A(H) + \det A(H \setminus x) \cdot \det A(G) \quad \blacksquare$$

The following two results are Rara's reduction formulas from [3]. They are immediate consequences of the theorem above.

Corollary 4.1.4. *Let x be a vertex of a graph G. If G^* is the graph obtained by adding to G a new vertex y together with the edge $\{x, y\}$(see the diagram), then*

$$\det A(G^*) = -\det A(G \setminus x)$$

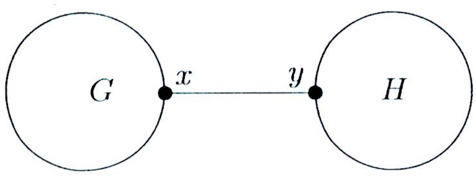


Proof. Take as H the graph consisting of (two vertices and) the edge $\{x, y\}$ and apply the above theorem. It suffices to notice, that $\det A(H) = -1$ and $\det A(H \backslash x) = 0$ as $H \backslash x$ is one element graph without edges. ∎

Corollary 4.1.5. *Suppose that G and H are two disjoint graphs and let F be the graph obtained by joining a vertex x of the graph G to a vertex y of H by a new edge(see the diagram below). Then*

$$\det A(F) = \det A(G) \cdot \det A(H) - \det A(G \backslash x) \cdot \det A(H \backslash y)$$

F :



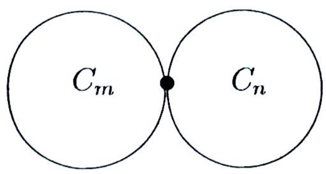
Proof. Let G^* be defined as in the above corollary with $x \in V(G)$ and $y \in V(H)$. Then, by Theorem 4.1.3, we get

$$\det A(G^* \cup H) = \det A(G) \cdot \det A(H) + \det A(H \backslash y) \cdot \det A(G^*)$$

$$= \det A(G) \cdot \det A(H) - \det A(H \backslash y) \cdot \det A(G \backslash x).$$ ∎

We can use Theorem 4.1.3 for computation of the determinant of the adjacency matrix for many graphs. For instance, suppose that G and H are cycles

F :



Then, by corollary 4.1.4 and corollary 4.1.5, we easily get

$$\det A(F) = \begin{cases} 0 & \text{if } m = 0 \text{ or } m = n = 2 \text{ or } m + 2 = n = 3; \\ -4 & \text{if } m + 1 = n = 2 \text{ or } m = n = 3 \\ 4 & \text{if } m + 1 = n = 3 \text{ or } m = n = 1. \end{cases}$$

where the equations $m = 0$ etc. are, clearly, meant modulo 4.

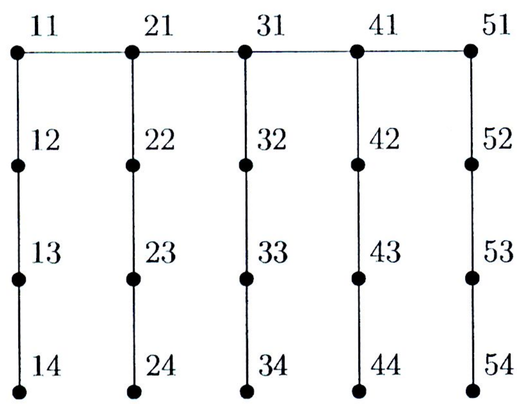
4.2 Path of Paths

In this section we compute the determinant of path of paths. We start by providing definition of the object.

Definition 4.2.1. Let m, n be positive integers such that $n \geq 1$. Let $P_n(P_m)$ be a graph with the vertex set $V(P_n(P_m)) = \{ij | i = 1, 2, ..., n \text{ and } j = 1, 2, ..., m\}$ and the edge set $E(P_n(P_m)) = \{\{ij, ik\} | \{j, k\} \in E(P_m) \cup \{\{i1, k1\} | \{i, k\} \in E(P_n)\}\}$. We call this graph the path P_n of paths P_m.

Example 4.2.2. The path P_5 of paths P_4.

$P_5(P_4)$:



Next, we recall that the determinant of path P_n.

Lemma 4.2.3. *Let P_n be a path of n vertices. Then*

$$\det(A(P_n)) = \begin{cases} (-1)^k & \text{if } n = 2k \text{ for some } k \in \mathbb{Z}^+, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.2.4. *Let $P_n(P_m)$ be a path, of the order n, of paths P_m, where $n \geq 1$ and $m \geq 2$. Then*

$$\det A(P_n(P_m)) = \begin{cases} 0 & \text{if } nm \text{ is odd;} \\ (-1)^{\frac{nm}{2}} & \text{if } nm \text{ is even.} \end{cases}$$

Proof. We prove our theorem by induction on n. If $n = 1$, it follows from corollary 4.1.5. Suppose that $n = 2$, then $P_n(P_m)$ is a path of order $2m$, so $\det A(P_n(P_m)) = (-1)^m$ which means that our theorem hold for $n = 2$.

Next, we suppose $n = k$ is true for all $m > 2$. Then we will show that it is true when $n = k + 1$. Then, by corollary 4.1.5

$\det A(P_n(P_m)) = \det A(P_k(P_m)) \cdot \det A(P_m) - \det A(P_{k-1}(P_m)) \cdot \det A(P_{m-1}) \cdot \det A(P_{m-1})$. If m is even, then $\det A(P_{m-1}) = 0$.and mn is even.

So $\det A(P_n(P_m)) = \det A(P_k(P_m)) \cdot \det A(P_m) = (-1)^{\frac{km}{2}}(-1)^{\frac{m}{2}} = (-1)^{\frac{km+m}{2}} = (-1)^{\frac{m(k+1)}{2}}$ which is what we need to prove. If m is odd, there are two possible cases.

$caseI$: $n = k + 1$ is even. Then, since $\det A(P_m) = 0$,

$$\begin{aligned} \det A(P_n(P_m)) &= -\det A(P_{k-1}(P_m)) \cdot [\det A(P_{m-1})]^2 \\ &= (-1)(-1)^{\frac{m(k-1)}{2}}(-1)^{\frac{2(m-1)}{2}} \\ &= (-1)^{\frac{mk-m+2}{2}} \cdot (-1)^{m-1} \\ &= (-1)^{\frac{m(k+1)}{2}} = (-1)^{\frac{mn}{2}}. \end{aligned}$$

$caseII$: $n = k + 1$ is odd. Then

$$\det A(P_n(P_m)) = -\det A(P_{k-1}(P_m)) \cdot [\det A(P_{m-1})]^2.$$

Since $k - 1$ is odd and m is odd, by the induction hypothesis, $\det A(P_{k-1}(P_m)) = 0$, $m(k-1)$ is odd. So $\det A(P_{k-1}(P_m)) = 0$, then $\det A(P_n(P_m)) = 0$.

∎

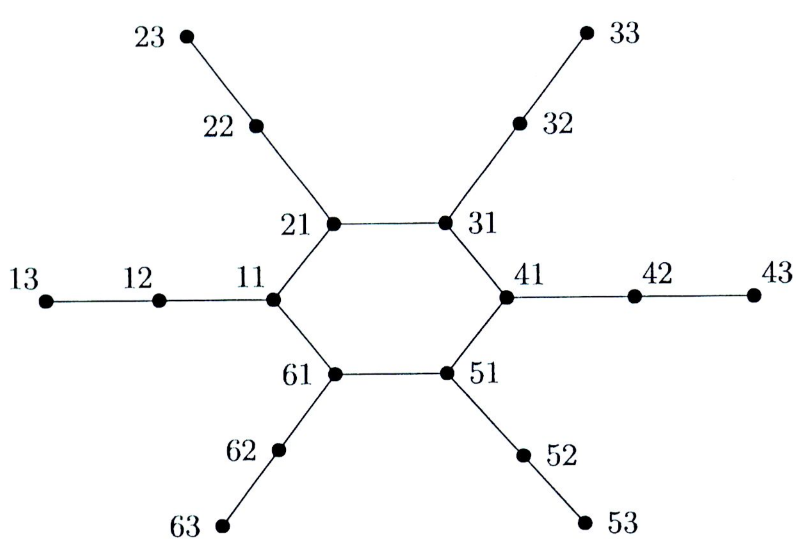
4.3 Cycle of Paths

In this section we compute the determinant of cycle of paths. We start by providing definition of the object.

Definition 4.3.1. Let m, n be positive integer such that $n \geq 3$. Let $C_n(P_m)$ be a graph with the vertex set $V(C_n(P_m)) = \{ij \mid i = 1, 2, ..., n \text{ and } j = 1, 2, ..., m\}$ and the edge set $E(C_n(P_m)) = \{\{ij, ik\} \mid \{j, k\} \in E(P_m)\} \cup \{\{i1, k1\} \mid \{i, k\} \in E(C_n)\}$ we call this graph, the cycle C_n of paths P_m.

Example 4.3.2. The cycle C_6 of paths P_3.

$C_6(P_3)$:



Next, we recall that the determinant of cycle C_n.

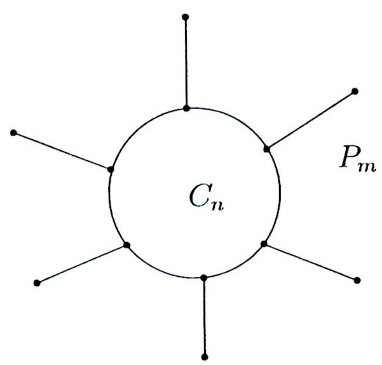
Lemma 4.3.3. *Let $n \geq 3$. Then*

$$\det(A(C_n)) = \begin{cases} 0 & \text{if } n \equiv 0 (mod 4) \text{ for some } k \in \mathbb{Z}^+, \\ -4 & \text{if } n \equiv 2 (mod 4) \\ 2 & \text{otherwise.} \end{cases}$$

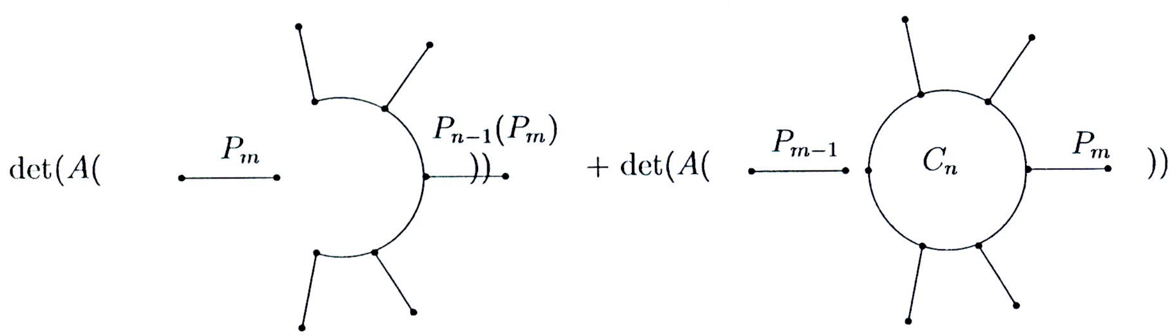
Theorem 4.3.4. *Let $n \geq 3$ and $m \geq 1$. Then*

$$\det A(C_n(P_m)) = \begin{cases} (-1)^{\frac{nm}{2}} & \textit{if } m \textit{ is even;} \\ 2(-1)^{\frac{nm-1}{2}} & \textit{if } n \textit{ and } m \textit{ is odd;} \\ 4(-1)^{\frac{nm-n+2}{2}} & \textit{if } m \textit{ is odd and } n \equiv 2 (mod 4); \\ 0 & \textit{if } m \textit{ is odd and } n \equiv 0 (mod 4). \end{cases}$$

Proof. We can apply Theorem 4.1.3 to derive, step by step paths P_m from a given necklace (see the picture below).



Denote the above graph by G_0. At the first step (of our iteration) we get



and $\det A(G_0) = \det A(P_m) \cdot \det A(P_{n-1}(P_m)) + \det A(P_{m-1}) \cdot \det A(G_1)$ where G_1, results from G_0 by the iteration of one of the paths P_m.

If m is even, then $\det A(P_{m-1}) = 0$.

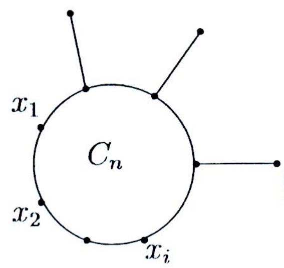
$$\text{So} \quad \det A(G_0) = \det A(P_m) \cdot \det A(P_{n-1}(P_m))$$
$$= (-1)^{\frac{m}{2}} \cdot (-1)^{\frac{m(n-1)}{2}}$$
$$= (-1)^{\frac{mn}{2}}.$$

If m is odd, then $\det A(P_{n-1}(P_m)) = 0$.

$$\text{So} \quad \det A(G_0) = \det A(P_{m-1}) \cdot \det A(G_1))$$

$$= (-1)^{\frac{m-1}{2}} \cdot \det A(G_1).$$

Next, we need to compute $\det A(G_1)$. We continue the iteration process in a similar fashion and at the $i-th$ step we get the following graph, denoted by G_i,



which there are lacking $i-th$ copies of P_m. We also get

$$\det A(G_{i-1}) = (-1)^{\frac{m-1}{2}} \cdot \det A(G_i).$$

Clearly, we have a sequence of graphs $G_0, G_1, ..., G_n$ where G_0 is the initial graph and G_n is C_n and G_i, $i < n$ is obtained from the original graph G_0 by deletion of $i-paths$. Thus

$$\det A(G_0) = (-1)^{\frac{m-1}{2}} \det A(G_1)$$

$$= (-1)^{\frac{m-1}{2}}[(-1)^{\frac{m-1}{2}} \det A(G_2)]$$

$$\vdots$$

$$= (-1)^{\frac{m-1}{2}} \cdot (-1)^{\frac{m-1}{2}} \cdots (-1)^{\frac{m-1}{2}} \cdot \det A(G_n)$$

$$= [(-1)^{\frac{m-1}{2}}]^n \cdot \det A(C_n).$$

Comparing this result with the formula for the derivative of a cycle we obtain the result of the theorem following

if n is odd, then $\det A(G_0) = 2(-1)^{\frac{nm-1}{2}}$

if $n \equiv 0(mod4)$, then $\det A(G_0) = 4(-1)^{\frac{nm-n+2}{2}}$

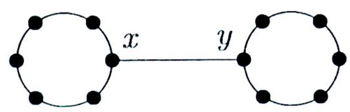
if $n \equiv 2(mod4)$, then $\det A(G_0) = 0$. ■

4.4 Path of Cycles

Theorem 4.4.1. *Let F be a path, of the order n, of cycles C_m, where $n \geq 1$ and $m \geq 3$. Then*

$$\det A(F) = \begin{cases} 0 & \text{if } m \equiv 0 (mod\ 4); \\ (-4)^n & \text{if } m \equiv 2 (mod\ 4); \\ n+1 & \text{otherwise.} \end{cases}$$

Proof. We prove our theorem by induction on n. If n =1, it follows immediately from Lemma 4.3.3. Suppose that $n = 2$. By Corollary 2.3.8, we get



$$\det A(F) = \det A(C_m) \cdot \det A(C_m) - \det A(C_m \backslash x) \cdot \det A(C_m \backslash y)$$

$$= \det A(C_m) \cdot \det A(C_m) - \det A(P_{m-1}) \cdot \det A(P_{m-1}).$$

If $m \equiv 0(mod4)$, then $\det A(C_m) = 0 = \det A(P_{m-1}) \cdot \det A(P_{m-1})$, and hence $\det A(F) = 0$ by Lemma 4.2.3 and Lemma 4.3.3.

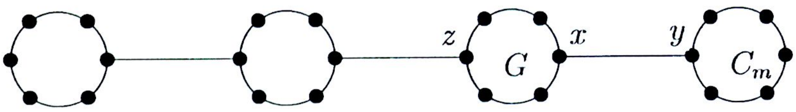
If $m \equiv 2(mod4)$, then $\det A(C_m) = -4$ and $\det A(P_{m-1}) = 0$, and hence $\det A(F) = (-4)^2 - 0 = 16$.

If m is odd, then $\det A(C_m) = 2$ and $\det A(P_{m-1}) = (-1)^{\frac{m-1}{2}}$, and hence $\det A(F) = 2^2 - 1 = 3$. Therefore,

$$\det A(F) = \begin{cases} 0 & \text{if } m \equiv 0(mod4); \\ 16 & \text{if } m \equiv 2(mod4); \\ 3 & \text{otherwise.} \end{cases}$$

Which means that our theorem holds for $n = 2$.

Suppose that F is a path, of the order $n > 2$, of cycles C_m and assume that our theorem holds for each $k-$path, with $k < n$, of cycles. Then we can claim that the graph F results by the "bridge" operation (see Corollary 2.3.8) applied to its subgraph G, which is a $(n-1)-$path of cycles, and a copy of C_m



Then, by Corollary 2.3.8,

$$\det A(F) = \det A(G) \cdot \det A(C_m) - \det A(G\backslash x) \cdot \det A(C_m\backslash y).$$

We know $\det A(G)$(by inductive hypothesis), and $\det A(C_m)$ (by Theorem 3.1.8), and $\det A(C_m\backslash y)$ (note that $C_m\backslash y$ is a path and apply Corollary 3.2.7). We do not know $\det A(G\backslash x)$, but it does not matter if m is even. Then
if $m \equiv 0(mod4)$, then $\det A(C_m) = \det A(P_{m-1}) = 0$ and hence we get $\det A(F) = 0$;
if $m \equiv 2(mod4)$, then $\det A(C_m) = -4$ and $\det A(G) = (-4)^{n-1}$; since $\det A(P_{m-1}) = 0$, we get $\det A(F) = \det A(C_m) \cdot \det A(G) = (-4)^n$.

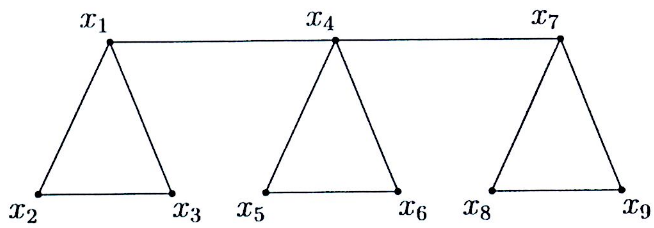
Suppose that m is odd. Then $\det A(P_{m-1}) = (-1)^{\frac{m-1}{2}}$, and $\det A(C_m) = 2$, and $\det A(G) = n$. There remains to compute $\det A(G\backslash x)$.

Note that the case with $x = z$ (see the diagram above) is not excluded. However, if it takes place, the computation of $\det A(G\backslash x)$ is relatively easy. Then H consists of two disjoint subgraphs. One of them is a path, of the order $n-2$, of cycles, and the second one is a copy of P_{m-1}. Then by Lemma 4.1.2, $\det A(G\backslash x) = (-1)^{\frac{m-1}{2}}(n-1)$ and hence $\det A(F) = 2n - (n-1) = n+1$. ■

Thus, we have computed the determinant of any path of similar cycles. There are many non-isomorphic paths (of order n) of $C_m's$ but, surprisingly, the determinant of the adjacency matrix of each of them is the same. The determinant depends on m and n, nothing else. In particular, it does not depend on the way in which the cycles are joined into a path(see the picture above). Our method of computing is more general . It could be applied, as well, to paths of cycles of different size. We have not introduced, however, such structures in our this work. Instead, let us illustrate the method by an example. This time we will refer to Theorem 4.1.3, instead of corollary 4.1.5, as in the proof of the above theorem, and each graph will represent the determinant of its adjacency matrix.

Example 4.4.2. Consider the following graph

$P_3(C_3)$:



$$\begin{aligned}
\det A(P_3(C_3)) &= \det A(P_3(C_3) \setminus \{x_4, x_7\}) + \det A(P_3(C_3) \setminus \{x_7, x_8\} \cup \{x_7, x_9\}) \\
&= \det A(P_3(C_3) \setminus \{x_4, x_7\}) + \det A(P_3(C_3) \setminus \{x_7, x_8\} \cup \{x_7, x_9\}) \\
&= (\det A(P_2))^3 \cdot \det A(C_3) + (\det A(C_3))^3 + \det A(C_3) \cdot (\det A(P_2))^3 \\
&= (-1)^3 \cdot 2 + (2)^3 + 2 \cdot (-1)^3 \\
&= 4.
\end{aligned}$$

4.5 Generalized Wheel Graphs

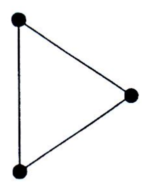
In this section we find the determinant of generalized wheel graph such that the definition following

Definition 4.5.1. Union of graphs G_1 and G_2 denote by $G_1 \bigcup G_2$ is graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

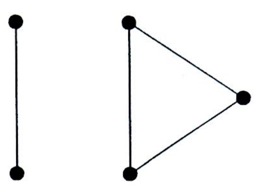
Definition 4.5.2. Sum of graphs G_1 and G_2 denote by $G_1 + G_2$ is a graph G such that $V(G) = V(G_1) \cup V(G_1)$ and$E(G) = E(G_1) \cup E(G_2) \cup \{v_1v_2/v_1 \in V(G_1)$ and $v_2 \in V(G_2)\}$.

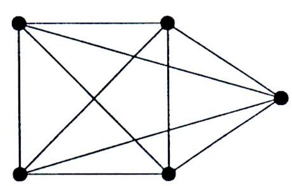
Example 4.5.3. Let G_1 and G_2 be graphs following.

G_1 : 

G_2 : 

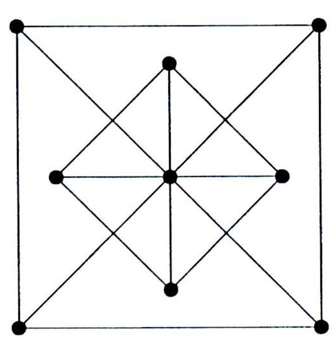
Then we get

$G_1 \bigcup G_2$: 

$G_1 + G_2$: 

Definition 4.5.4. Let m, n be positive integers such that $m \geq 1$and $n \geq 3$. Then generalized wheel graph, denote by $W_n(m)$, is a graph $\underbrace{(C_n \bigcup C_n \bigcup ... \bigcup C_n)}_{m} + C_1$.

Example 4.5.5. $W_4(2)$



In [1], Iyad T. Abu-Jeib found the determinant of the adjacency matrices of the wheel graphs $det(A(W_n))$ by using the multiplication of their eigenvalues. His result is

$$\det(A(W_n)) = \begin{cases} 2n & \text{if } n \equiv 2(mod 4) \\ 0 & \text{if } n \equiv 0(mod 4) \\ -n & \text{otherwise.} \end{cases}$$

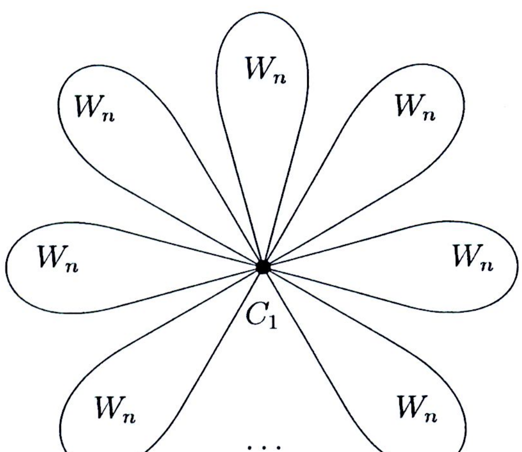
From result above and theorem 4.1.3, we can find the determinant of $W_n(m)$ following.

Theorem 4.5.6. *Let $m \geq 2$ and $n \geq 3$. Then*

$$\det A(W_n(m)) = \begin{cases} (m-2)(-4)^{m-1}2n - (-4)^m n & \textit{if } n \equiv 2(mod\ 4); \\ 0 & \textit{if } n \equiv 0(mod\ 4); \\ (m-2)(-n)(2)^{m-1} + (-n)2^m & \textit{otherwise.} \end{cases}$$

Proof. We can considering graph of $W_n(m)$ is a graph G_0 in form following

G_0 :



So $\det A(G_0) = \det A(C_n) \det A(G_1) + (\det A(C_n))^{m-1} \det A(W_n(m))$. We can consider following if $n \equiv 2(mod)4$, then

$$\begin{aligned} \det A(G_0) &= (-4) \det A(G_1) + (-4)^{2m} 2n \\ &= (-4)^{m-1}2n + (-4)[\det A(C_n) \det A(G_2) + (\det A(C_n))^{m-1} \det A(W_n)] \\ &= (-4)^{m-1}2n + (-4)^{m-1}2n + 16 \det A(G_2) \\ &= (-4)^{m-1}2n + (-4)^{m-1}2n + 16[(-4) \det A(G_3) + (-4)^{m-3}2n] \\ &= (-4)^{m-1}2n + (-4)^{m-1}2n + (-4)^2[(-4) \det A(G_3) + (-4)^{m-3}2n] \end{aligned}$$

$$
\begin{aligned}
&= (-4)^{m-1}2n + (-4)^{m-1}2n + (-4)^{m-1}2n - 64 \det A(G_3) \\
&\quad\vdots \\
&= (m-2)(-4)^{m-1}2n + (-4)^{m-2} \det A(G_{m-2}) \\
&= (m-2)(-4)^{m-1}2n - (-4)^{m-2}16n \\
&= (m-2)(-4)^{m-1}2n - (-4)^{m}n.
\end{aligned}
$$

If $n \equiv (0 mod 4)$, then $\det A(C_n) = 0$ so we see that

$$
\begin{aligned}
\det A(G_0) &= \det A(C_n) \det A(G_1) + (\det A(C_n))^{m-1} \det A(W_n \\
&= (0) \det A(G_1) + (0) \det A(W_n(m)) \\
&= 0.
\end{aligned}
$$

If $n \equiv (1, 3 \; mod \; n)$, then

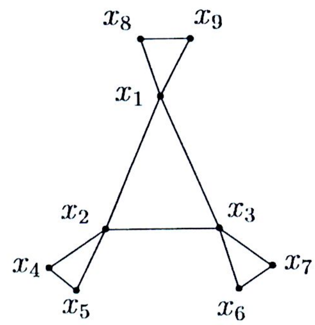
$$
\begin{aligned}
\det A(G_0) &= \det A(C_n) \det A(G_1) + (\det A(C_n))^{m-1} \det A(W_n) \\
&= (2) \det A(G_1) + (2)^{m-1} \det A(W_n) \\
&= (2)[2 \det A(G_2) + (2)^{m-2}(-n)] + (-n)2^{m-1} \\
&= 4 \det A(G_2) + (-n)2^{m-1} + (-n)(2)^{m-1} \\
&= 4[2 \det A(G_3) + (-n)2^{m-3}] + (-n)(2)^{m-1} + (-n)2^{m-1} \\
&= (-n)2^{m-1} + (-n)2^{m-1} + (-n)2^{m-1} + 8 \det A(G_3) \\
&\quad\vdots \\
&= (m-2)(-n)(2)^{m-1} + 2^{m-2} \det A(G_{m-2}) \\
&= (m-2)(-n)(2)^{m-1} + (-n)2^{m}.
\end{aligned}
$$

■

4.6 Cycle of Cycles

In this section we will try to compute the determinant of any necklace graph $C_n(C_m)$. To perform this, we will derive, step by step, any cycle from the necklace using Theorem 4.1.3. First, let us illustrate our method by means of some examples. Consider $C_3(C_3)$,

$C_3(C_3)$:

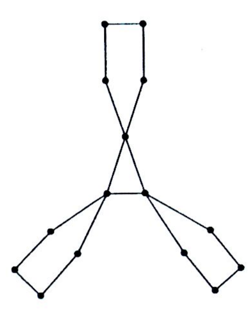


we get

$$\begin{aligned}\det A(C_3(C_3)) &= \det A(C_3(C_3)\backslash\{x_1,x_8\}\cup\{x_1,x_9\})+\det A(C_3(C_3)\backslash\{x_1,x_2\}\cup\{x_1,x_3\}) \\ &= (\det A(C_3 \cup C_3)\cdot(\det A(P_2))^2) + (\det A(P_2))^3 \cdot \det A(C_3) \\ &\quad + \det A(P_2(C_3)) \cdot \det A(C_3) \\ &= (2)(-1) + (-1)(2) + (-1)^3(2) + (3)(2) \\ &= -2 - 2 - 2 + 6 = 0.\end{aligned}$$

Let us observe that the numbers we get for cycles of cycles may be different from those given for paths of cycles. For instance, it follows from Theorem 4.4.1 that $\det A(P_3(C_3)) = 4 \neq 0 = \det A(C_3(C_3))$. However, we can also show, in a similar way as above, that $\det A(C_3(C_5)) = 4 = \det A(P_3(C_5))$.

$C_3(C_5)$:

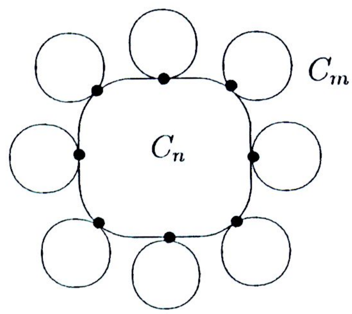


Now, let us prove the main theorem of this section.

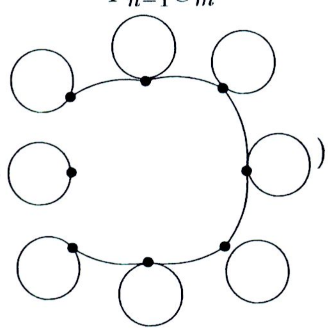
Theorem 4.6.1. *Let $n, m \geq 3$. Then*

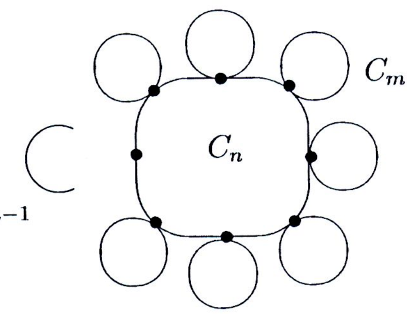
$$\det A(C_n(C_m)) = \begin{cases} (-4)^n & \text{if } m \equiv 2 (mod 4); \\ 4 & \text{if } m \equiv 1 (mod 4) \text{ and } n \text{ is odd}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We apply Theorem 4.1.3 to derive, step by step, cycles C_m from a given necklace (see the picture below).



Denote the above graph by G_0. At the first step(of our derivation) we get

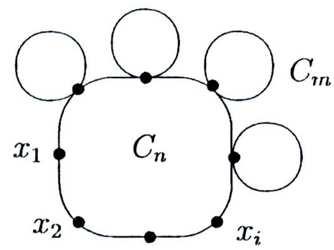
det $A($ C_m 

$)$ + det $A($ P_{m-1} 

$)$

and $\det A(G_0) = \det A(C_m) \cdot \det A(P_{n-1}(C_m)) + \det A(P_{m-1}) \cdot \det A(G_1)$ where G_1 results from G_0 by the derivation of one of the cycles C_m. If m is even, $\det A(P_{m-1}) = 0$ by Lemma 4.2.3 and hence $\det A(G_0) = 0$ if $m \equiv 0(mod4)$ or $\det A(G_0) = (-4)^n$ if $m \equiv 2(mod4)$, by Corollary 4.1.4 and Theorem 4.4.1. Thus, we can assume that m is odd and hence $\det A(P_{m-1}) = (-1)^{\frac{m-1}{2}}$ and $\det A(C_m) = 2$. Then $\det A(G_0) = 2 \cdot \det A(P_{n-1}(C_m)) + (-1)^{\frac{m-1}{2}} \cdot \det A(G_1)$ and we need to compute $\det A(G_1)$.

We continue the derivation of cycles and at the $i-th$ step we get the following graph, denote by G_i



in which there are lacking $i-th$ copies of C_m. We also get

$$\det A(G_{i-1}) = 2\cdot \det A(G_i\backslash x_i) + (-1)^{\frac{m-1}{2}}\cdot \det A(G_i)$$

In result, we obtain a sequence of graph $G_0, G_1, ..., G_n$ where G_0 is the initial necklace graph and G_n is C_n. By the above formula we could compute $\det A(G_0)$ if we know $\det A(G_i\backslash x_i)$. But using Corollary 3.6, we get

$$\det A(G_i\backslash x_i) = \begin{cases} (-1)^{\frac{i-1}{2}}\cdot \det A(P_{n-i}(C_m)) & \text{if } i \text{ is odd;}\\ (-1)^{\frac{i}{2}}\cdot \det A(P_{m-1})\cdot \det A(P_{n-i-1}(C_m)) & \text{if } i \text{ is even.}\end{cases}$$

This means (see Theorem 4.1) that

$$\det A(G_i\backslash x_i) = \begin{cases} (-1)^{\frac{i-1}{2}}(n-i+1) & \text{if } i \text{ is odd;}\\ (-1)^{\frac{m+i-1}{2}}(n-i) & \text{if } i \text{ is even.}\end{cases}$$

Suppose that $m\equiv 1(mod4)$. Then we have

$$\det A(G_{i-1}) = (-1)^{\lfloor\frac{i}{2}\rfloor}\cdot 2\cdot(n-2\cdot\lfloor\frac{i}{2}\rfloor + \det A(G_i))$$

and hence

$$\begin{aligned}\det A(G_0) &= 2n + \det A(G_1)\\ &= 2n - 2(n-2) + \det A(G_2)\\ &= 2n-2(n-2)-2(n-2)+\det A(G_3)\\ &= 2n-2(n-2)-2(n-2)+2(n-4)+\det A(G_4)\\ &= \sum_{i=1}^{n}(-1)^{\lfloor\frac{i}{2}\rfloor}\cdot 2\cdot(n-2\cdot\lfloor\frac{i}{2}\rfloor) + \det A(G_n).\end{aligned}$$

Thus, our theorem holds for $m\equiv 1(mod4)$ as $G_n = C_n$(see Corollary 3.4) and

$$\sum_{i=1}^{n}(-1)^{\lfloor\frac{i}{2}\rfloor}\cdot 2\cdot(n-2\cdot\lfloor\frac{i}{2}\rfloor)=\begin{cases}0 & \text{if } n\equiv 0(mod4);\\ 4 & \text{if } n\equiv 2(mod4);\\ 2 & \text{otherwise.}\end{cases}$$

Hence $\det A(G_0) = 4$ if n is odd, and $\det A(G_0) = 0$ otherwise.

Suppose that $m \equiv 3(mod4)$. Our argumentation is similar as above.

$$\det A(G_{i-1}) = (-1)^{\lfloor\frac{i}{2}\rfloor}\cdot 2\cdot(n-2\cdot\lfloor\frac{i}{2}\rfloor - \det A(G_i))$$

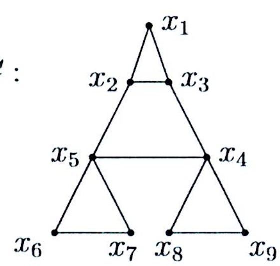
and hence

$$\begin{aligned}\det A(G_0) &= 2n - \det A(G_1)\\ &= 2n-2(n-2)+\det A(G_2)\\ &= 2n-2(n-2)-2(n-2)-\det A(G_3)\\ &= 2n-2(n-2)-2(n-2)+2(n-4)+\det A(G_4)\\ &= \sum_{i=1}^{n}(-1)^{\lfloor\frac{i}{2}\rfloor}\cdot 2\cdot(n-2\cdot\lfloor\frac{i}{2}\rfloor)+(-1)^n\det A(G_n)\end{aligned}$$

which means that $\det A(G_0) = 0$ as required. ∎

In contrast to paths of cycles, the determinant of the adjacency matrix of any cycle of cycles depends on the way in which the cycles are joined into a cycle. The above theorem concerns only necklace graphs and it is not true for arbitrary cycles of cycles. For instance, we know that $C_3(C_3)$ is singular whereas the following graph

$G:$



which is a cycle, of the order 3, of $C_3's$ is not singular. We have

$$\begin{aligned}\det A(G) &= \det A(G\backslash\{x_4, x_8\} \cup \{x_4, x_9\}) \cdot \det A(P_2) + \det A(P_2(C_3)) \cdot \det A(C_3)\\ &= \det A(G\backslash\{x_5, x_6\} \cup \{x_5, x_7\} \cup \{x_4, x_8\} \cup \{x_4, x_9\})\\ &\quad + (\det A(P_2))^2 \cdot \det A(C_3) + \det A(P_2(C_3)) \cdot \det A(C_3)\\ &= 0 + 2 + (3)(2)\\ &= 8.\end{aligned}$$

In the last line of the above calculation we refer to Theorem 4.1.2. It is still open if one can define a feasible algorithm for solving the problem of singularity of any cycle of cycles, e.g.