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On Solving Centrosymmetric Linear Systems

Moawwad El-Mikkawy[*], Faiz Atlan
Mathematics Department, Faculty of Science, Mansoura University, Mansoura, Egypt
Email: [*]m_elmikkawy@yahoo.com, faizatlan11@yahoo.com

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ABSTRACT

The current paper is mainly devoted for solving centrosymmetric linear systems of equations. Formulae for the determinants of tridiagonal centrosymmetric matrices are obtained explicitly. Two efficient computational algorithms are established for solving general centrosymmetric linear systems. Based on these algorithms, a MAPLE procedure is written. Some illustrative examples are given.

Keywords: Centrosymmetric Matrix; Exchange Matrix; Rotate Matrix; Linear System; DETGTRI Algorithm; Computer Algebra Systems (CAS); MAPLE

1. Introduction

Throughout this paper, $|A|$ and A^{T} denote the determinant and the transpose of the matrix A respectively. Also $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Centrosymmetric matrices have practical applications in numerical analysis, information theory, statistics, physics, harmonic differential quadrature, differential equations, engineering, sinc methods, magic squares, linear system theory and pattern recognition. The interested readers may refer to [1-12].

Solving and analyzing linear systems of equations is a fundamental problem in science and engineering applications. The cost of solving any linear system using Gauss or Gauss-Gordan algorithms is $O(n^3)$. The motivation of the current paper is to develop efficient algorithms for solving any centrosymmetric linear system having n equations and n unknowns provided that the coefficient matrix of the system is nonsingular. The cost of each algorithm depends on the solvers of two associated linear systems having smaller sizes than n. More precisely, if $n = 2m$, then each of the two associated linear systems consists of m equations. If $n = 2m + 1$, then we have one system having m equations and the other has $(m + 1)$ equations. Consequently, if the two associated linear systems have special structures, then the cost of the centrosymmetric algorithm could be considerably reduced, in particular for large values of n.

The paper is organized as follows. In Section 2, some properties of the exchange and the rotate matrices are presented. Formulae for centrosymmetric tridiagonal determinants are obtained in Section 3. In Section 4, two computational algorithms for solving centrosymmetric linear systems are given. A MAPLE procedure is given in Section 5. Some illustrative examples are presented in Section 6.

Definition 1.1. The $n \times n$ matrix with 1's on the northeast-southeast diagonal and 0's elsewhere

$$J_n = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \tag{1}$$

is called the exchange matrix of order n. The subscript on J_n is neglected whenever the size is obvious from the context.

Definition 1.2. Let $A = (a_{ij})_{i,j=1}^{n}$ be an $n \times n$ matrix. The rotate of A, denoted A^R, is defined by

$$A^R = JAJ. \tag{2}$$

Definition 1.3 [9]. Let $A = (a_{ij})_{i,j=1}^{n}$ be an $n \times n$ matrix. Then

- A is said to be centrosymmetric if $A^R = A$.
- A is said to be persymmetric if $A^R = A^{\mathrm{T}}$.

[*]Corresponding author.

- A is said to be centrogonal if $A^R = A^{-1}$, provided $|A| \neq 0$.
- A is said to be skew-centrosymmetric if $A^R = -A$.
- A is said to be bisymmetric if $A^R = A = A^{\mathrm{T}}$.

Note that the bisymmetric matrix is both symmetric and centrosymmetric. It is also both symmetric and pre-symmetric.

2. Some Properties of The Exchange and The Rotate Matrices

Let us begin this section by giving some helpful results concerning the exchange and rotate matrices. For more details see [1,3,8,9,12-21].

The exchange matrix J_n enjoys the following properties:

- $J = J^{\mathrm{T}} = J^{-1}$.
- $J_n^k = \begin{cases} I_n, & k \text{ even} \\ J_n, & k \text{ odd}, \end{cases}$

where I_n is the identity matrix of order n.

- $|J_n| = (-1)^{\frac{n(n-1)}{2}} = \begin{cases} 1, & \text{if } n \equiv 0 \text{ or } 1 \bmod (4) \\ -1, & \text{if } n \equiv 2 \text{ or } 3 \bmod (4). \end{cases}$
- The matrix product JA is a version of the matrix A that has been reflected in line at 0 degree to the horizontal measured counter-clockwise.
- The matrix product AJ is a version of the matrix A that has been reflected in line at 90 degrees to the horizontal measured counter-clockwise.
- The matrix product $JAJ = A^R$ is a version of the matrix A that has been rotated counter-clockwise or clockwise by 180 degrees.

Note that the exchange matrix J is equal to the identity matrix I but with the columns in reverse order. More precisely, $J = (e_n, e_{n-1}, \cdots, e_1)$ where $e_i = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{ni})$, $i = 1, 2, \cdots, n$, δ_{ij} is the Kronecker symbol which is equal to 1 if $i = j$ and zero if $i \neq j$.

It is also worth mentioned that the matrix J is sometimes called the counter-identity matrix or contra-identity matrix or per-identity matrix or the reflection matrix or the reversal matrix.

Exchange matrices are simple to construct in software platforms. For example, to construct J_5 in MAPLE, a single line of code can be used as follows:

n := 5: *J* := array(1..*n*,1..*n*,sparse): for *i* to *n* do *J*[*i*, *n* + 1 - *i*] := 1 od: *J*|| *n* := op(*J*);

or *n* := 5: *J* := matrix(*n*, *n*, 0): for *i* to *n* do *J*[*i*, n + 1 - *i*] := 1 od: *J*|| *n* := op(*J*);

The rotate of A, A^R of order n satisfies:

- $\left(A^R\right)^R = A$.
- $\left(A^R\right)^{\mathrm{T}} = \left(A^{\mathrm{T}}\right)^R$.
- $\left(A^{-1}\right)^R = \left(A^R\right)^{-1}$, provided A^{-1} exists.
- $(AB)^R = A^R B^R$.
- $(A \pm B)^R = \left(A^R \pm B^R\right)$.
- $\left|A^R\right| = |A|$.
- If row_i of A is $\left(a_{i1}, a_{i2}, \cdots, a_{i,n-1}, a_{in}\right)$, then row_{n+1-i} of A^R is $\left(a_{in}, a_{i,n-1}, \cdots, a_{i2}, a_{i1}\right)$. In other words, the centrosymmetric matrix A is the same when read backwards as when read forwards.

3. Centrosymmetric Determinants

Centrosymmetric determinants take the form:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{2,n} & \cdots & a_{2,2} & a_{2,1} \\ a_{1,n} & \cdots & a_{1,2} & a_{1,1} \end{vmatrix}, \qquad (3)$$

in which

$$row_i = \left(a_{i1}, a_{i2}, \cdots, a_{i,n-1}, a_{in}\right),$$

and

$$row_{n+1-i} = \left(a_{in}, a_{i,n-1}, \cdots, a_{i,2}, a_{i1}\right),$$

for each $i = 1, 2, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$.

In particular, centrosymmetric tridiagonal determinants are of special importance. For convenience of the reader, we present some definitions, notations and properties associated with tridiagonal matrices.

A tridiagonal matrix $T = \left(a_{ij}\right)_{i,j=1}^n$ takes the form:

$$T = \begin{bmatrix} d_1 & a_1 & 0 & \cdots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-1} & d_{n-1} & a_{n-1} \\ 0 & \cdots & 0 & b_n & d_n \end{bmatrix}. \qquad (4)$$

These types of matrices frequently appear in many areas of science and engineering. For example in parallel computing, telecommunication system analysis and in solving differential equations using finite differences, see [22,23]. A general $n \times n$ tridiagonal matrix of the form (4) can be stored in $3n$ memory locations, rather than n^2 memory locations for a full matrix, by using three vectors $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$, $\boldsymbol{b} = (b_1, b_2, \cdots, b_n)$, and $\boldsymbol{d} = (d_1, d_2, \cdots, d_n)$ with $b_1 = a_n = 0$. This is always a good habit in computation in order to save memory space. To study tridiagonal matrices it is very convenient to introduce a vector $\boldsymbol{c}$ in the following way [24]:

$$\boldsymbol{c} = (c_1, c_2, \cdots, c_n), \qquad (5)$$

where

$$c_1 = d_1, \quad c_k = d_k - \frac{b_k a_{k-1}}{c_{k-1}}, \quad k = 2,3,\cdots,n. \tag{6}$$

It is helpful to restate some important results concerning tridiagonal matrices of the form (4). For more details the interested reader may refer to [24-28].

Let p, q, r and s are positive integers such that $1 \le p \le q \le n$ and $1 \le r \le s \le n.$

Define the submatrix of $A,$ denoted $A[p..q, r..s],$ of order $(q-p+1)\times(s-r+1)$ by

$$A[p..q, r..s] = \begin{bmatrix} a_{p,r} & a_{p,r+1} & \cdots & a_{p,s} \\ a_{p+1,r} & a_{p+1,r+1} & \cdots & a_{p+1,s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q,r} & a_{q,r+1} & \cdots & a_{q,s} \end{bmatrix}. \tag{7}$$

In particular, let $F_k = A[1..k, 1..k],$ $1 \le k \le n,$ and $f_k = |F_k|,$ the leading principal minor of $A.$

Theorem 3.1 [24]. Consider

$$\begin{aligned} f_1 &= \left|T[1..1, 1..1]\right| = |d_1| = d_1, \\ f_k &= \left|T[1..k, 1..k]\right| = |F_k|, k = 2,3,\cdots,n. \end{aligned} \tag{8}$$

Then the determinants in (8) satisfy a three-term recurrence

$$f_k = d_k f_{k-1} - b_k a_{k-1} f_{k-2}, k = 1,2,\cdots,n, \tag{9}$$

where the initial values for f_k are $f_0 = 1$ and $f_{-1} = 0.$

Lemma 3.2 [29]. If the LU factorization of the matrix T in (4) is possible, then we have

$$|T| = \prod_{k=1}^{n} c_k. \tag{10}$$

where $c_1, c_2, \cdots, c_n$ are given by (6). Meanwhile, the Doolittle LU factorization [30] of T is given by:

$T = LU,$ where

$$L = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \frac{b_2}{c_1} & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \frac{b_{n-1}}{c_{n-2}} & 1 & 0 \\ 0 & \cdots & 0 & \frac{b_n}{c_{n-1}} & 1 \end{bmatrix},$$

$$\text{and } U = \begin{bmatrix} c_1 & a_1 & 0 & \cdots & 0 \\ 0 & c_2 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & c_{n-1} & a_{n-1} \\ 0 & \cdots & \cdots & 0 & c_n \end{bmatrix}. \tag{11}$$

Lemma 3.3 [26]. If $c_r \ne 0, r = 1,2,\cdots,n,$ then

$$f_i = \prod_{k=1}^{i} c_k, i = 1,2,\cdots,n. \tag{12}$$

Lemma 3.4 [26]. If $c_i = 0,$ $1 \le i \le n-1,$ and either $a_i = 0$ or $b_{i+1} = 0,$ then T is a singular matrix.

Lemma 3.5 [24]. If $c_i \ne 0$ for each $i = 1,2,\cdots,n-1,$ then the three-term recurrence (9) reduces to the two-term recurrence

$$f_i = c_i f_{i-1}, \quad i = 1,2,\cdots,n. \tag{13}$$

Algorithm 3.1 (DETGTRI [31]).

The determinant of the matrix in (4) can be computed using the following symbolic algorithm.

INPUT: Order of the matrix n and the components, $a_i, d_i, b_i, i = 1,2,\cdots,n, (a_n = b_1 = 0).$

OUTPUT: The determinant of the matrix T in (4).

Step 1: Use (6) to compute the simplest forms of the n components of the vector $\mathbf{c}$.

If $c_i = 0$ for any $i \le n$, set $c_i = t$ (t is just a symbolic name) and continue to compute $c_{i+1}, c_{i+2}, \cdots, c_n$ in terms of t by using (6).

Step 2: The simplest rational form of the product $P(t) = \prod_{r=1}^{n} c_r$ (this product is a polynomial in t) evaluated at $t = 0$ is equal to the determinant of the matrix T in (4), *i.e.*, $|T| = P(0).$

The cost of the **DETGTRI** algorithm is $O(n)$. The algorithm is easy to implement in all Computer Algebra Systems (CAS) such as MACSYMA, MATHEMATICA and MAPLE.

Lemma 3.6. Consider the tridiagonal matrix T_n given by:

$$T_n = \begin{bmatrix} d_1 & a_1 & 0 & \cdots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-1} & d_{n-1} & a_{n-1} \\ 0 & \cdots & 0 & b_n & d_n \end{bmatrix}. \tag{14}$$

Let $S_n,$ U_n and V_n be $n \times n$ matrices defined respectively as follows:

$$\begin{aligned} & S_n = T_n\big|_{d_n \to d_n \pm k}, \quad U_n = T_n\big|_{a_{n-1} \to k a_{n-1}} \\ & \text{and} \quad V_n = T_n\big|_{b_n \to k b_n}, \end{aligned} \tag{15}$$

where k is a scalar quantity. Then by applying the **DETGTRI** algorithm, we see that:

$$|S_n| = |T_n| \pm k|T_{n-1}|, \tag{16}$$

$$|U_n| = |V_n| = |T_n| - (k-1) b_n a_{n-1} |T_{n-2}|, \tag{17}$$

having used (12) and (13).

Let $A = (a_{ij})_{i,j=1}^{n}$ is an $n \times n$ centrosymmetric matrix, then the three following facts are useful when we deal with such matrices:

Fact (1): If $row_i = \left(a_{i1}, a_{i2}, \cdots, a_{i,n-1}, a_{in}\right)$, then

$$row_{n+1-i} = \left(a_{in}, a_{i,n-1}, \cdots, a_{i2}, a_{i1}\right).$$

In other words, the centrosymmetric matrix A is the same when read backwards as when read forwards.

Fact (2): If $n = 2m$ is positive even number, then

$$Q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} I_m & I_m \\ J_m & -J_m \end{bmatrix},$$

is orthogonal.

Fact (3): If $n = 2m+1$ is positive odd number then

$$Q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} I_m & 0 & I_m \\ 0 & \sqrt{2} & 0 \\ J_m & 0 & -J_m \end{bmatrix},$$

is orthogonal.

Armed with the above facts, we may formulate the following result whose proof will be omitted.

Theorem 3.7. Let $R = \left(r_{ij}\right)_{i,j=1}^{n}$ be an $n \times n$ of even order say $n = 2m$, then R can be written in the form:

$$R = \begin{bmatrix} A & JBJ \\ B & JAJ \end{bmatrix} = Q_1 \begin{bmatrix} A+JB & \\ & A-JB \end{bmatrix} Q_1^{\mathrm{T}}, \qquad (18)$$

where $A, B \in \mathbb{R}^{m\times m}$.

The determinant of the matrix R in (18) is given by:

$$|R| = |A+JB||A-JB|. \qquad (19)$$

If n is odd, *i.e.*, $n = 2m+1$, then we have:

$$R = \begin{bmatrix} A & \boldsymbol{v} & JBJ \\ \boldsymbol{u}^{\mathrm{T}} & q & \boldsymbol{u}^{\mathrm{T}}J \\ B & J\boldsymbol{v} & JAJ \end{bmatrix} = Q_2 \begin{bmatrix} A+JB & \sqrt{2}\boldsymbol{v} & 0 \\ \sqrt{2}\boldsymbol{u}^{\mathrm{T}} & q & 0 \\ 0 & 0 & A-JB \end{bmatrix} Q_2^{\mathrm{T}}, \qquad (20)$$

where $A, B \in \mathbb{R}^{m\times m}$, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^{m\times 1}$ and $q \in \mathbb{R}$.

The determinant of the matrix R in (20) is given by

$$|R| = \begin{vmatrix} A+JB & \sqrt{2}\boldsymbol{v} \\ \sqrt{2}\boldsymbol{u}^{\mathrm{T}} & q \end{vmatrix} |A-JB|. \qquad (21)$$

Concerning the inverse of centrosymmetric matrices the reader may refer to [8].

As an interesting special case of the Theorem 3.7, we give the following result.

Corollary 3.8. In Theorem 3.7, if $R = T_n$ is centrosymmetric tridiagonal matrix, then we have

$$|T_n| = \begin{cases} |T_m|^2 - k^2 |T_{m-1}|^2, & \text{if } n = 2m, \\ |T_m|\left(|T_{m+1}| - b_{m+1}a_m |T_{m-1}|\right), & \text{if } n = 2m+1. \end{cases}$$

where k is the common value of the elements in positions $(m, m+1)$ and $(m+1, m)$ of the matrix $R = T_n$ when $n = 2m$.

Proof. To compute the centrosymmetric tridiagonal determinant of order n. Two cases will be considered:

Case (1): $n = 2m$. In this case, the centrosymmetric tridiagonal matrix takes the form:

$$T_n = T_{2m} = \left[\begin{array}{ccccc|ccccc} d_1 & a_1 & 0 & \cdots & 0 & 0 & \cdots & & \cdots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots & \vdots & & & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & & & & & \\ \vdots & \ddots & b_{m-1} & d_{m-1} & a_{m-1} & 0 & & & & \vdots \\ 0 & \cdots & 0 & b_m & d_m & k & 0 & & \cdots & 0 \\ \hline 0 & \cdots & & 0 & k & d_m & b_m & 0 & \cdots & 0 \\ \vdots & & & & 0 & a_{m-1} & d_{m-1} & b_{m-1} & \ddots & \vdots \\ & & & & & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \vdots & \vdots & \ddots & a_2 & d_2 & b_2 \\ 0 & \cdots & & \cdots & 0 & 0 & \cdots & 0 & a_1 & d_1 \end{array}\right]. \qquad (22)$$

Applying Theorem 3.7, we have $|T_n| = |T_{2m}| = |L||M|$, where

$$L = A + JB = \begin{bmatrix} d_1 & a_1 & 0 & \cdots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{m-1} & d_{m-1} & a_{m-1} \\ 0 & \cdots & 0 & b_m & d_m + k \end{bmatrix}$$

and

$$M = A - JB = \begin{bmatrix} d_1 & a_1 & 0 & \cdots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{m-1} & d_{m-1} & a_{m-1} \\ 0 & \cdots & 0 & b_m & d_m - k \end{bmatrix}.$$

By using Lemma 3.6, we obtain

$$|L| = |T_m| + k|T_{m-1}|$$

and

$$|M| = |T_m| - k|T_{m-1}|.$$

Therefore we get

$$\begin{aligned} |T_n| &= |T_{2m}| = |L||M| \\ &= |T_m|^2 - k^2|T_{m-1}|^2 . \end{aligned} \tag{23}$$

Case (2): $n = 2m+1$. In this case, the centrosymmetric tridiagonal matrix takes the form:

$$T_n = T_{2m+1} = \left[\begin{array}{ccccc|c|ccccc} d_1 & a_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & & \cdots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots & \vdots & \vdots & & & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & \vdots & & & & & \\ \vdots & \ddots & b_{m-1} & d_{m-1} & a_{m-1} & 0 & \vdots & & & & \vdots \\ 0 & \cdots & 0 & b_m & d_m & a_m & 0 & \cdots & & \cdots & 0 \\ \hline 0 & \cdots & \cdots & 0 & b_{m+1} & d_{m+1} & b_{m+1} & 0 & \cdots & & 0 \\ \hline 0 & \cdots & & \cdots & 0 & a_m & d_m & b_m & 0 & \cdots & 0 \\ \vdots & & & & \vdots & 0 & a_{m-1} & d_{m-1} & b_{m-1} & \ddots & \vdots \\ & & & & & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \vdots & \vdots & \vdots & \ddots & a_2 & d_2 & b_2 \\ 0 & \cdots & & \cdots & 0 & 0 & 0 & \cdots & 0 & a_1 & d_1 \end{array}\right]. \tag{24}$$

Applying Theorem 3.7, we obtain $|T_n| = |T_{2m+1}| = |L||M|$, here

$$L = A - JB = A = T_m = \begin{bmatrix} d_1 & a_1 & 0 & \cdots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{m-1} & d_{m-1} & a_{m-1} \\ 0 & \cdots & 0 & b_m & d_m \end{bmatrix}$$

and

$$M = \begin{bmatrix} A + JB & \sqrt{2}\boldsymbol{v} \\ \sqrt{2}\boldsymbol{u}^{\mathrm{T}} & q \end{bmatrix} = \begin{bmatrix} d_1 & a_1 & 0 & \cdots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_m & d_m & \sqrt{2}a_m \\ 0 & \cdots & 0 & \sqrt{2}b_{m+1} & d_{m+1} \end{bmatrix}.$$

By using Lemma 3.6, we obtain $|L| = |T_m|$ and

$$|M| = \begin{vmatrix} A + JB & \sqrt{2}\boldsymbol{v} \\ \sqrt{2}\boldsymbol{u}^{\mathrm{T}} & q \end{vmatrix} = \begin{vmatrix} d_1 & a_1 & 0 & \cdots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_m & d_m & \sqrt{2}a_m \\ 0 & \cdots & 0 & \sqrt{2}b_{m+1} & d_{m+1} \end{vmatrix}.$$

By using the **DETGTRI** algorithm [31] together with (12) and (13), then we have

$$\left|M\right| = \begin{vmatrix} d_1 & a_1 & 0 & \cdots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_m & d_m & 2a_m \\ 0 & \cdots & 0 & b_{m+1} & d_{m+1} \end{vmatrix} = \begin{vmatrix} d_1 & a_1 & 0 & \cdots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_m & d_m & a_m \\ 0 & \cdots & 0 & 2b_{m+1} & d_{m+1} \end{vmatrix}$$

$$= \left(c_1 c_2 \cdots c_m\right)\alpha_{m+1} = \left|T_m\right|\alpha_{m+1} = \left|T_m\right|\left(d_{m+1} - 2\frac{b_{m+1}a_m}{c_m}\right) = \left|T_m\right|\left(c_{m+1} - \frac{b_{m+1}a_m}{c_m}\right)$$

$$= c_{m+1}\left|T_m\right| - \frac{\left|T_m\right|}{c_m}b_{m+1}a_m = \left|T_{m+1}\right| - \frac{\left|T_m\right|}{c_m}b_{m+1}a_m = \left|T_{m+1}\right| - b_{m+1}a_m\left|T_{m-1}\right|.$$

Therefore

$$\begin{aligned} \left|T_n\right| &= \left|T_{2m+1}\right| \\ &= \left|T_m\right|\left(\left|T_{m+1}\right| - b_{m+1}a_m\left|T_{m-1}\right|\right). \end{aligned} \tag{25}$$

From (23) and (25), we see that in order to compute the determinant of a centrosymmetric tridiagonal matrix of order n, then all we need is to compute $c_1, c_2, \cdots, c_m$ if $n = 2m$ and $c_1, c_2, \cdots, c_{m+1}$ if $n = 2m+1$.

4. Algorithms for Solving Centrosymmetric Linear Systems

Solving linear systems practically dominates scientific computing. In the present section, we focus on solving linear systems of centrosymmetric type. Two cases will be considered:

Case (i): $n = 2m$. For this case we are going to construct an algorithm for solving centrosymmetric linear systems of the form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,m} & a_{1,m+1} & \cdots & a_{1,2m-1} & a_{1,2m} \\ a_{21} & a_{22} & \cdots & a_{2,m} & a_{2,m+1} & \cdots & a_{2,2m-1} & a_{2,2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m,m} & a_{m,m+1} & \cdots & a_{m,2m-1} & a_{m,2m} \\ a_{m,2m} & a_{m,2m-1} & \cdots & a_{m,m+1} & a_{m,m} & \cdots & a_{m,2} & a_{m,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2,2m} & a_{2,2m-1} & \cdots & a_{2,m+1} & a_{2,m} & \cdots & a_{22} & a_{21} \\ a_{1,2m} & a_{1,2m-1} & \cdots & a_{1,m+1} & a_{1,m} & \cdots & a_{12} & a_{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ \vdots \\ x_{2m-1} \\ x_{2m} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ \vdots \\ b_{2m-1} \\ b_{2m} \end{bmatrix}. \tag{26}$$

Block multiplication is particularly useful when there are patterns in the matrices to be multiplied. Therefore it is convenient to rewrite (26) in the partitioned form

$$\left[\begin{array}{c|c} A & JBJ \\ \hline B & JAJ \end{array}\right]\left[\begin{array}{c} X_1 \\ \hline X_2 \end{array}\right] = \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right], \tag{27}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,m} \\ a_{21} & a_{22} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m,m} \end{bmatrix} \quad B = \begin{bmatrix} a_{m,2m} & a_{m,2m-1} & \cdots & a_{m,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{2,2m} & a_{2,2m-1} & \cdots & a_{2,m+1} \\ a_{1,2m} & a_{1,2m-1} & \cdots & a_{1,m+1} \end{bmatrix},$$

$$X_1 = \left[x_1, x_2, \cdots, x_m\right]^{\mathrm{T}},\ X_2 = \left[x_{m+1}, x_{m+2}, \cdots, x_{2m}\right]^{\mathrm{T}},$$

$$B_1 = \left[b_1, b_2, \cdots, b_m\right]^{\mathrm{T}} \text{ and } B_2 = \left[b_{m+1}, b_{m+2}, \cdots, b_{2m}\right]^{\mathrm{T}}.$$

The system in (26) can also be written in matrix form as follows:

$$R\boldsymbol{x} = \boldsymbol{b}, \tag{28}$$

where $R = \left(a_{ij}\right)_{i,j=1}^{2m}$ is the coefficient matrix of the system (26), $\boldsymbol{x} = \left[x_1, x_2, \cdots, x_{2m}\right]^{\mathrm{T}}$ and $\boldsymbol{b} = \left[b_1, b_2, \cdots, b_{2m}\right]^{\mathrm{T}}$

is the constant vector.

Algorithm 4.1. An algorithm for solving centrosymmetric linear system of even order.

To solve the linear system of the form (26), we may proceed as follows:

INPUT: The entries of the coefficient matrix R and the constant vector $\boldsymbol{b}$ in (28).

OUTPUT: Solution vector $\boldsymbol{x} = \left[x_1, x_2, \cdots, x_{2m}\right]^{\mathrm{T}}$.

Step 1: Construct the $m \times m$ matrices P, Q and the m-vectors $\hat{\boldsymbol{b}}$ and $\tilde{\boldsymbol{b}}$ as follows:

$$P = A + JB = \left[a_{ij} + a_{i,2m+1-j}\right]_{i,j=1}^{m},$$

$$Q = A - JB = \left[a_{ij} - a_{i,2m+1-j}\right]_{i,j=1}^{m},$$

$$\hat{\boldsymbol{b}} = \left[b_1 + b_{2m}, b_2 + b_{2m-1}, \cdots, b_m + b_{m+1}\right]^{\mathrm{T}}$$

and

$$\tilde{\boldsymbol{b}} = \left[b_1 - b_{2m}, b_2 - b_{2m-1}, \cdots, b_m - b_{m+1}\right]^{\mathrm{T}}.$$

Step 2: Compute $|R| = |P||Q|$. If $|R| = 0$ then Exiterror('No solutions') end if.

Step 3: Solve the two linear systems:

$P\boldsymbol{y} = \hat{\boldsymbol{b}}$, and $Q\boldsymbol{z} = \tilde{\boldsymbol{b}}$, for $\boldsymbol{y} = \left[y_1, y_2, \cdots, y_m\right]^{\mathrm{T}}$ and $\boldsymbol{z} = \left[z_1, z_2, \cdots, z_m\right]^{\mathrm{T}}$ respectively.

Step 4: The solution vector $\boldsymbol{x} = \left[x_1, x_2, \cdots, x_{2m}\right]^{\mathrm{T}}$ is given by

$$x_i = \begin{cases} \frac{1}{2}\left(y_i + z_i\right) & \text{if } i = 1, 2, \cdots, m, \\ \frac{1}{2}\left(y_{2m+1-i} - z_{2m+1-i}\right) & \text{if } i = m+1, m+2, \cdots, 2m. \end{cases}$$

The Algorithm 4.1, will be refereed to as **CENTRO-SYMM-I** algorithm.

Concerning the computational cost of the **CENTRO-SYMM-I** algorithm:

- The time complexity of **Step 1** is

$$\begin{aligned} & m^2 + m^2 + m + m \\ & = 2m^2 + 2m = 2 \times \left(\frac{n}{2}\right)^2 + 2 \times \frac{n}{2} \\ & = \frac{1}{2}n^2 + n = \frac{1}{2}n^2 + O(n) \end{aligned}$$

($\frac{1}{2}n^2 + n$ additions/subtractions and no multiplications/divisions).

- The time complexity of **Step 3** depends on the solvers of the two linear systems. For example, tridiagonal linear systems can be solved in linear time (see [25]).
- The time complexity of **Step 4** is $O(n)$ (n additions/subtractions and n multiplications).

Step 3 is the step that leads to the reduction of the time complexity, because instead of solving a linear system of n equations, we end up with two linear systems half the size of the original one. If the original system is solved with Gaussian elimination (GE) method, then the time complexity will be $\frac{2}{3}n^3 + O\left(n^2\right)$. But, if GE method is used to solve the two systems in the third step, then the time complexity of our algorithm will be

$$\frac{4}{3}m^3 + O\left(m^2\right) = \frac{1}{6}n^3 + O\left(n^2\right),$$

which is a significant reduction. If a method more efficient than the GE method is used, then the time complexity of our algorithm will be less (see also [16]).

Case (ii): $n = 2m+1$. In this case the linear system to be considered has the form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,m} & a_{1,m+1} & a_{1,m+2} & \cdots & a_{1,2m} & a_{1,2m+1} \\ a_{21} & a_{22} & \cdots & a_{2,m} & a_{2,m+1} & a_{2,m+2} & \cdots & a_{2,2m} & a_{2,2m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m,m} & a_{m,m+1} & a_{m,m+2} & \cdots & a_{m,2m} & a_{m,2m+1} \\ a_{m+1,1} & a_{m+1,2} & \cdots & a_{m+1,m} & a_{m+1,m+1} & a_{m+1,m} & \cdots & a_{m+1,2} & a_{m+1,1} \\ a_{m,2m+1} & a_{m,2m} & \cdots & a_{m,m+2} & a_{m,m+1} & a_{m,m} & \cdots & a_{m,2} & a_{m,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2,2m+1} & a_{2,2m} & \cdots & a_{2,m+2} & a_{2,m+1} & a_{2,m} & \cdots & a_{22} & a_{21} \\ a_{1,2m+1} & a_{1,2m} & \cdots & a_{1,m+2} & a_{1,m+1} & a_{1,m} & \cdots & a_{12} & a_{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m+1} \\ \vdots \\ x_{2m} \\ x_{2m+1} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m+1} \\ \vdots \\ b_{2m} \\ b_{2m+1} \end{bmatrix}, \tag{29}$$

or equivalently,

$$\left[\begin{array}{c|c|c} A & \boldsymbol{v} & JBJ \\ \hline \boldsymbol{u}^{\mathrm{T}} & q & \boldsymbol{u}^{\mathrm{T}}J \\ \hline B & J\boldsymbol{v} & JAJ \end{array}\right] \left[\begin{array}{c} X_1 \\ \hline x_{m+1} \\ \hline X_2 \end{array}\right] = \left[\begin{array}{c} B_1 \\ \hline b_{m+1} \\ \hline B_2 \end{array}\right], \tag{30}$$

where

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,m} \\ a_{21} & a_{22} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m,m} \end{bmatrix} \quad B=\begin{bmatrix} a_{m,2m+1} & a_{m,2m} & \cdots & a_{m,m+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{2,2m+1} & a_{2,2m} & \cdots & a_{2,m+2} \\ a_{1,2m+1} & a_{1,2m} & \cdots & a_{1,m+2} \end{bmatrix},$$

$$\boldsymbol{v}=\left[a_{1,m+1},a_{2,m+1},\cdots,a_{m,m+1}\right]^{\mathrm{T}}, \quad \boldsymbol{u}=\left[a_{m+1,1},a_{m+1,2},\cdots,a_{m+1,m}\right]^{\mathrm{T}}, \quad q=\left[a_{m+1,m+1}\right],$$

$$X_1=\left[x_1,x_2,\cdots,x_m\right]^{\mathrm{T}}, \quad X_2=\left[x_{m+2},x_{m+3},\cdots,x_{2m+1}\right]^{\mathrm{T}},$$

$$B_1=\left[b_1,b_2,\cdots,b_m\right]^{\mathrm{T}} \text{ and } B_2=\left[b_{m+2},b_{m+3},\cdots,b_{2m+1}\right]^{\mathrm{T}}.$$

The linear system (30) can also be written as:

$$R\boldsymbol{x}=\boldsymbol{b}, \tag{31}$$

where $R=\left(a_{ij}\right)_{i,j=1}^{2m+1}$ is the coefficient matrix of the system (29),

$$\boldsymbol{x}=\left[x_1,x_2,\cdots,x_{2m+1}\right]^{\mathrm{T}} \text{ and } \boldsymbol{b}=\left[b_1,b_2,\cdots,b_{2m+1}\right]^{\mathrm{T}}$$

is the constant vector.

Algorithm 4.2. An algorithm for solving centrosymmetric linear system of odd order

To solve the linear system of the form (29), we may proceed as follows:

INPUT: The entries of the coefficient matrix R and the constant vector $\boldsymbol{b}$ in (31).

OUTPUT: Solution vector $\boldsymbol{x}=\left[x_1,x_2,\cdots,x_{2m+1}\right]^{\mathrm{T}}$.

Step 1: Construct the matrices P, Q of orders $m+1$ and m respectively and the vectors $\hat{\boldsymbol{b}}$ and $\tilde{\boldsymbol{b}}$ of dimensions $m+1$ and m respectively as follows:

$$P=\begin{bmatrix} A+JB & 2\boldsymbol{v} \\ \boldsymbol{u}^{\mathrm{T}} & q \end{bmatrix}=\begin{bmatrix} a_{1,1}+a_{1,2m+1} & a_{1,2}+a_{1,2m} & \cdots & a_{1,m}+a_{1,m+2} & 2a_{1,m+1} \\ a_{2,1}+a_{2,2m+1} & a_{2,2}+a_{2,2m} & \cdots & a_{2,m}+a_{2,m+2} & 2a_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1}+a_{m,2m+1} & a_{m,2}+a_{m,2m} & \cdots & a_{m,m}+a_{m,m+2} & 2a_{m,m+1} \\ a_{m+1,1} & a_{m+1,2} & \cdots & a_{m+1,m} & a_{m+1,m+1} \end{bmatrix},$$

$$Q=A-JB=\left[a_{ij}-a_{i,2m+2-j}\right]_{i,j=1}^{m},$$

$$\hat{\boldsymbol{b}}=\left[b_1+b_{2m+1},b_2+b_{2m},\cdots,b_m+b_{m+2},b_{m+1}\right]^{\mathrm{T}}$$

and

$$\tilde{\boldsymbol{b}}=\left[b_1-b_{2m+1},b_2-b_{2m},\cdots,b_m-b_{m+2}\right]^{\mathrm{T}}.$$

Step 2: Compute $|R|=|P||Q|$, If $|R|=0$ then Exiterror(“No solutions”) end if.

Step 3: Solve the two linear systems $P\boldsymbol{y}=\hat{\boldsymbol{b}}$, and $Q\boldsymbol{z}=\tilde{\boldsymbol{b}}$, for

$$\boldsymbol{y}=\left[y_1,y_2,\cdots,y_m,y_{m+1}\right]^{\mathrm{T}} \text{ and } \boldsymbol{z}=\left[z_1,z_2,\cdots,z_m\right]^{\mathrm{T}}$$

respectively.

Step 4: The solution vector $\boldsymbol{x}=\left[x_1,x_2,\cdots,x_{2m+1}\right]^{\mathrm{T}}$ is given by

$$x_i=\begin{cases} \dfrac{1}{2}\left(y_i+z_i\right) & \text{if } i=1,2,\cdots,m, \\ y_{m+1} & \text{if } i=m+1, \\ \dfrac{1}{2}\left(y_{2m+2-i}-z_{2m+2-i}\right) & \text{if } i=m+2,m+3,\cdots,2m+1. \end{cases}$$

The Algorithm will be refereed to as **CENTRO-SYMM-II** algorithm.

Concerning the computational cost of the **CENTRO-SYMM-II** algorithm:

- The time complexity of **Step 1** is $\frac{1}{2}n^2+O(n)$.

- The time complexity of **Step 3** depends on the solvers of the two linear systems.
- The time complexity of **Step 4** is $O(n)$ (see also [16]).

It may be convenient to finish this section by giving the following result, whose proof will be omitted.

Theorem 4.1. Let $A=\left(a_{ij}\right)_{i,j=1}^{n}$ be a non-singular centrosymmetric square matrix of order *n.* Consider the four linear systems of centrosymmetric type:

$$A\boldsymbol{x}=b, \tag{32}$$

$$AJ\boldsymbol{x}=b, \tag{33}$$

$$JA\boldsymbol{x}=b \tag{34}$$

and

$$A^{R}\boldsymbol{x}=b. \tag{35}$$

Then the two linear systems (32) and (35) are equivalent. The same is true for the linear systems (33) and (34). Moreover, if the common solution of the systems (32) and (35) is $\boldsymbol{x}=\left(k_1,k_2,\cdots,k_n\right)^{\mathrm{T}}$, then the common solution of the systems (33) and (34) is $\boldsymbol{x}=\left(k_n,k_{n-1},\cdots,k_1\right)^{\mathrm{T}}$.

5. Computer Program

In this section, we are going to introduce a MAPLE procedure for solving centrosymmetric linear systems (26) and (29). This procedure is based on the **CENTROSYMM-I** and **CENTROSYMM-II** Algorithms.

```
 > restart:
   centrosymm:=proc(R::array,f::vector,n::posint)
          local i, r,m,f1,f2,A,Jm,J,H,y,x,B,Y,Z,X:
          global xsoln,detR,detP,detQ,P,Q;
          X:= vector(n): m:=floor(n/2): J:=array(1..m, 1..m,
sparse):
          for i to m do J[m+1-i,i]:=1 od:
          A:= linalg[submatrix](R,1..m,1..m):
          if n=2*m then
            #  Case(1): n is even  #
            #  Step 1 in CENTROSYMM-I Algorithm.
            B:= linalg[submatrix](R,m+1..n,1..m):
            P:=evalm( A + evalm(J&*B)):
            Q:=evalm( A - evalm(J&*B)):
            #  Step 2 in CENTROSYMM-I Algorithm.
            detP:=linalg[det](P): detQ:=linalg[det](Q):
            detR:=detP *detQ:
            if detR = 0 then ERROR("Singular Matrix !!!! ")
            fi;
            #  Step 3 in CENTROSYMM-I Algorithm.
            f1:=array(1..m): f2:=array(1..m):
            for i to m do
                f1[i]:=f[i]+f[2*m+1-i];
                f2[i]:=f[i]-f[2*m+1-i];
            od;
            Y:=array(1..m): Z:=array(1..m):
            Y:=  linalg[linsolve](P,f1):   Z:=linalg[linsolve]
            (Q,f2):
            #  Step 4 in CENTROSYMM-I Algorithm.
            for i to m do
               X[i]:=1/2*(Y[i]+Z[i]);
            od;
            for i from m+1 to n do
               X[i]:=1/2*(Y[n+1-i]-Z[n+1-i]);
            od;
            xsoln:=simplify([seq(X[r],r=1..n)]):
         else
            # Case(2): n is odd  #
            #  Step 1 in CENTROSYMM-II Algorithm.
            B:= linalg[submatrix](R,m+2..n,1..m):
            H := evalm( A + evalm(J&*B)):
            y:=linalg[submatrix](R,1..m,m+1..m+1):
            x:=linalg[submatrix](R,m+1..m+1,1..m):
            P:=linalg[blockmatrix](2,2,[H,2*y,x,[2]]):op(P);
            Q:=evalm( A - evalm(J&*B)):
            #  Step 2 in CENTROSYMM-II Algorithm.
            detP:=linalg[det](P):  detQ:=linalg[det](Q):  detR:=
            detP *detQ:
            if detR = 0 then ERROR("Singular Matrix !!!!")
            fi;
            #  Step 3 in CENTROSYMM-II Algorithm.
            f1:=array(1..m+1): f2:=array(1..m):
            for i to m do
               f1[i]:=f[i]+f[2*m+2-i];
               f2[i]:=f[i]-f[2*m+2-i];
            od;
            f1[m+1]:=f[m+1];
            Y:=array(1..m+1):
            Y:=  linalg[linsolve](P,f1):   Z:=linalg[linsolve]
            (Q,f2):
            #  Step 4 in CENTROSYMM-II Algorithm.
            for i to m do
               X[i]:=1/2*(Y[i]+Z[i]);
            od;
            X[m+1]:=Y[m+1];
            for i from m+2 to n do
               X[i]:=1/2*(Y[n+1-i]-Z[n+1-i]);
            od;
            xsoln:=simplify([seq(X[r],r=1..n)]);
         fi:
end proc:
```

6. Illustrative Examples

All results in this section are obtained with the help of the MAPLE procedure **centrosymm**.

Example 6.1. Solve the centrosymmetric linear system

$$\begin{bmatrix} 3 & -1 & 1 & 0 & 4 & 1 & -3 & 2 & -3 & 1 \\ 2 & -1 & -3 & 4 & -2 & -1 & -1 & 0 & 3 & 2 \\ 1 & 0 & 2 & -2 & 3 & 3 & 2 & 1 & 3 & 1 \\ 3 & 3 & -3 & 4 & -1 & 4 & -2 & 3 & -1 & 1 \\ 1 & 1 & 2 & -1 & 3 & 2 & 1 & -1 & 3 & 0 \\ 0 & 3 & -1 & 1 & 2 & 3 & -1 & 2 & 1 & 1 \\ 1 & -1 & 3 & -2 & 4 & -1 & 4 & -3 & 3 & 3 \\ 1 & 3 & 1 & 2 & 3 & 3 & -2 & 2 & 0 & 1 \\ 2 & 3 & 0 & -1 & -1 & -2 & 4 & -3 & -1 & 2 \\ 1 & -3 & 2 & -3 & 1 & 4 & 0 & 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \\ 9 \\ 16 \\ 6 \\ 9 \\ 4 \\ 7 \\ 2 \\ 6 \end{bmatrix}.$$

Solution:
Here $n = 10$ and $m = 5$. Using the procedure **centrosymm**, we get the following results:

$$P = \begin{bmatrix} 4 & -4 & 3 & -3 & 5 \\ 4 & 2 & -3 & 3 & -3 \\ 2 & 3 & 3 & 0 & 6 \\ 4 & 2 & 0 & 2 & 3 \\ 1 & 4 & 1 & 0 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 2 & -1 & 3 & 3 \\ 0 & -4 & -3 & 5 & -1 \\ 0 & -3 & 1 & -4 & 0 \\ 2 & 4 & -6 & 6 & -5 \\ 1 & -2 & 3 & -2 & 1 \end{bmatrix},$$

$$\hat{\boldsymbol{b}} = [5, 13, 16, 20, 15]^{\mathrm{T}} \text{ and } \tilde{\boldsymbol{b}} = [-7, 9, 2, 12, -3]^{\mathrm{T}}, \quad |R| = |P| \times |Q| = -730912.$$

The solutions of the systems

$$\begin{bmatrix} 4 & -4 & 3 & -3 & 5 \\ 4 & 2 & -3 & 3 & -3 \\ 2 & 3 & 3 & 0 & 6 \\ 4 & 2 & 0 & 2 & 3 \\ 1 & 4 & 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 16 \\ 20 \\ 15 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 2 & -1 & 3 & 3 \\ 0 & -4 & -3 & 5 & -1 \\ 0 & -3 & 1 & -4 & 0 \\ 2 & 4 & -6 & 6 & -5 \\ 1 & -2 & 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 2 \\ 12 \\ -3 \end{bmatrix}$$

are $\boldsymbol{y} = [2, 1, -1, 2, 2]^{\mathrm{T}}$ and $\boldsymbol{z} = [0, -1, -1, 0, -2]^{\mathrm{T}}$.

Therefore, we have

$$x_1 = \frac{1}{2}(y_1 + z_1) = 1, \quad x_6 = \frac{1}{2}(y_5 - z_5) = 2,$$

$$x_2 = \frac{1}{2}(y_2 + z_2) = 0, \quad x_7 = \frac{1}{2}(y_4 - z_4) = 1,$$

$$x_3 = \frac{1}{2}(y_3 + z_3) = -1, \quad x_8 = \frac{1}{2}(y_3 - z_3) = 0,$$

$$x_4 = \frac{1}{2}(y_4 + z_4) = 1, \quad x_9 = \frac{1}{2}(y_2 - z_2) = 1,$$

$$x_5 = \frac{1}{2}(y_5 + z_5) = 0, \quad x_{10} = \frac{1}{2}(y_1 - z_1) = 1.$$

Hence the solution vector is

$$\boldsymbol{x} = [1, 0, -1, 1, 0, 2, 1, 0, 1, 1]^{\mathrm{T}}.$$

Example 6.2. Solve the centrosymmetric linear system

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \\ 4 \\ 3 \\ 1 \\ 1 \\ 3 \\ 4 \\ 3 \end{bmatrix}.$$

Solution: Here $n = 11$ and $m = 5$. Using the procedure **centrosymm**, we obtain the following results:

$$P=\begin{bmatrix}2&1&0&0&0&0\\1&2&1&0&0&0\\0&1&2&1&0&0\\0&0&1&2&1&0\\0&0&0&1&2&2\\0&0&0&0&1&2\end{bmatrix},\quad Q=\begin{bmatrix}2&1&0&0&0\\1&2&1&0&0\\0&1&2&1&0\\0&0&1&2&1\\0&0&0&1&2\end{bmatrix},$$

$\hat{\boldsymbol{b}}=[5,5,4,4,5,3]^{\mathrm{T}}$ and $\tilde{\boldsymbol{b}}=[-1,-3,-2,2,3]^{\mathrm{T}}$,

$|R|=|P|\times|Q|=12.$

The solutions of the systems

$$\begin{bmatrix}2&1&0&0&0&0\\1&2&1&0&0&0\\0&1&2&1&0&0\\0&0&1&2&1&0\\0&0&0&1&2&2\\0&0&0&0&1&2\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\\y_5\\y_6\end{bmatrix}=\begin{bmatrix}5\\5\\4\\4\\5\\3\end{bmatrix}$$

and

$$\begin{bmatrix}2&1&0&0&0\\1&2&1&0&0\\0&1&2&1&0\\0&0&1&2&1\\0&0&0&1&2\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\\z_4\\z_5\end{bmatrix}=\begin{bmatrix}-1\\-3\\-2\\2\\3\end{bmatrix}$$

are $\boldsymbol{y}=[2,1,1,1,1,1]^{\mathrm{T}}$ and $\boldsymbol{z}=[0,-1,-1,1,1]^{\mathrm{T}}$.

Therefore, we get

$$x_1=\frac{1}{2}(y_1+z_1)=1,\quad x_7=\frac{1}{2}(y_5-z_5)=0,$$

$$x_2=\frac{1}{2}(y_2+z_2)=0,\quad x_8=\frac{1}{2}(y_4-z_4)=0.$$

$$x_3=\frac{1}{2}(y_3+z_3)=0,\quad x_9=\frac{1}{2}(y_3-z_3)=1.$$

$$x_4=\frac{1}{2}(y_4+z_4)=1,\quad x_{10}=\frac{1}{2}(y_2-z_2)=1.$$

$$x_5=\frac{1}{2}(y_5+z_5)=1,\quad x_{11}=\frac{1}{2}(y_1-z_1)=1.$$

$$x_6=y_6=1,$$

Hence the solution vector is

$$\boldsymbol{x}=[1,0,0,1,1,1,0,0,1,1,1]^{\mathrm{T}}.$$

Example 6.3. Solve the centrosymmetric linear system

$$\begin{bmatrix}1&0&-1&2&3&-2&1&0&4&1\\2&1&3&-1&6&6&-1&3&1&2\\3&2&1&0&-1&2&1&1&3&4\\4&3&2&1&0&1&3&4&0&1\\5&4&3&0&1&2&4&5&1&3\\3&1&5&2&2&1&2&3&4&5\\1&0&4&3&1&0&1&2&3&4\\4&3&1&1&2&-1&0&1&2&3\\2&1&3&-1&6&6&-1&3&1&2\\1&4&0&1&-2&3&2&-1&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\\x_8\\x_9\\x_{10}\end{bmatrix}=\begin{bmatrix}-1\\2\\1\\3\\0\\3\\1\\1\\4\\3\end{bmatrix}.$$

Solution:

Here $n=10$ and $m=5$. Using the procedure **centrosymm**, we obtain the following results:

Error Singular Matrix !!!! This means that the given system has no solutions.

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